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Magneto-hydrodynamic Shock Structure Without Collisions

by

Cathleen S. Morawetz

March 31, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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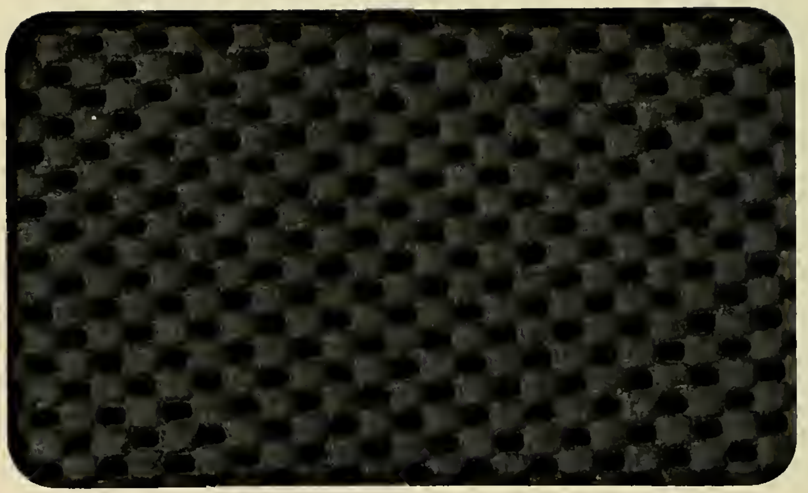

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ERRATA

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1) page 15, line 9 from bottom, read "dt" for "dy".
2) page 18, line 4 from bottom, read "(23a)" for "(23aa)".
3) page 34, line 8 from bottom, read "any plane $x = \tilde{x}$ with $|x_* - \tilde{x}| = o(1/\epsilon)$ at a time $t = \tilde{t}$".
4) page 35, line 7, read "$\lim \epsilon\tilde{v} \leq 0$".
5) page 36, line 6 from bottom, "see Lemma A1, in Appendix 3".
6) page 38, line 7, read "positive value of u" for "positive value u".
7) page 39. In the second figure the axes are x and u.
8) page 45. In the figure, the curve cuts axis orthogonally at $x = x_1$.
9) page 55, line 5 from bottom, read "$r^2 = [(u-1)^2 + v^2]_{P_\infty}$".

ABSTRACT

In this report we present a method for studying the internal structure of a magneto-hydrodynamic shock in which, in contrast to earlier theories, there are no collisions among the plasma particles. The theory yields us two distinct situations. In the first case, there is no shock. The ions and electrons pass through a region of changing magnetic and electric field and eventually they return to the state from which they started. In the second case, there is a definite change of state. Within the approximation we obtain a qualitative picture of the shock. The magnetic field rises and then oscillates with a characteristic wave length about a mean value greater than its initial value. The distribution function of the ions is greatly distorted. The approximation is based on an asymptotic development in the ion to electron mass ratio. It is valid over distances that are comparable to or even large compared to the wave length of the oscillation but small compared to the ion Larmor radius. Thus it is quite possible that an improvement in the range of validity of the approximation would show that the oscillation is damped out (e.g., on a length scale comparable to the ion Larmor radius) and that a classical shock exists. However, the change of state shows that an irreversible process has been demonstrated to be possible without collisions, and this already marks the difference between existence of shocks and nonexistence.

It should be noted that the fields and flows are self-consistent within the approximation.

The results of this paper were partially described in [13] and also in [7] and [8].

The author is grateful to C. S. Gardner for stimulating discussions about and invaluable contributions to this paper, and to K. O. Friedrichs for his thorough critical study of the manuscript.

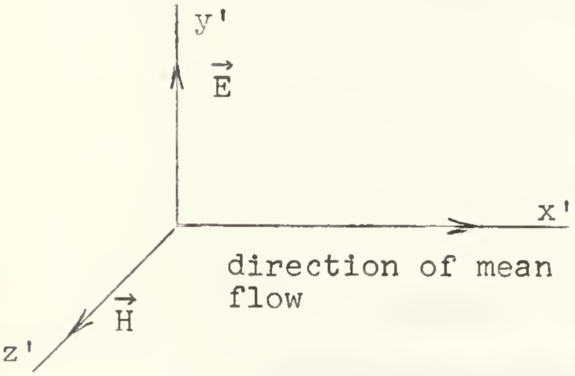
MAGNETO HYDRODYNAMIC SHOCK STRUCTURE WITHOUT COLLISIONS

1. General Introduction.

To determine the internal structure of a magneto-hydrodynamic shock one wants to establish the existence of a steady state solution of the one-dimensional equations of motion for two fluids and Maxwell's equations, with the property that it leads from a given constant state to a constant state at higher pressure. If the second state is the same as the first we call the solution a pulse. This problem has been under study for some time. Early theories, see [11], [12], [14], [17], all involved in some sense, collisions, either through viscosity, thermal conductivity or electric conductivity. In those cases where a transition stage leads from one constant state to another the length of this state is rather long — of the order of a mean free path. This distance may actually, in certain cases, compare with the dimensions of the apparatus. For a useful shock theory one needs a formulation that does not involve collisions. This is a radical departure from the usual procedure and it was first proposed by H. Grad, see [7] and [9], that such a theory was possible. Also such a shock theory can, in contrast to a collision theory, lead to a difference in the temperature of ions and electrons.

We begin by assuming that there are no collisions and do not treat the gas as a continuous fluid. We seek a flow and field that are self-consistent; this means that the influence of ions and electrons on electric and magnetic fields and vice versa is taken into account*. Properly speaking, we seek a time independent one-dimensional solution of two Boltzmann equations (one for ions and one for electrons) and of the Maxwell equations for the electromagnetic field.

If x' is the space variable, the conditions at $x' = -\infty$ are that the fields are given constants and the velocity distributions of the ions and electrons are also given. We consider the following situation: the electric field is in the y'-direction and the magnetic field is in the z'-direction. The distribution functions f_+ for the ions and f_- for the electrons are prescribed to be compatible with a constant state. As we shall show, we can work with distribution functions at $-\infty$, $f_\pm^\infty$ which depend only on u', v', the velocity components in the x and y directions. We take the distribution function for the ions at $-\infty$ as a function of $(U-u')^2 + v'^2$ where U is the mean velocity at $-\infty$. Since the flow is assumed to be steady, U is also the shock speed.



In order to simplify the problem we take $f_+^\infty = 0$ for $(U-u')^2 + v'^2 > R'^2$. We refer to R' as the cut-off radius. Depending on how we choose the cut-off radius we can observe different effects.

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[*] A totally different approach to this problem involving "turbulence" in the magnetic field is given in [6]. A fluid approach involving strong shocks is given in [4].

The fields at $-\infty$ are restricted to be those for which the Alfven-Mach number $1 < U/a < \nu < 2$ where a is Alfven speed and ν is some number depending on the distribution at $-\infty$. Its value must be determined in each case by numerical computations.

The basis for approximation is that $\epsilon^2 = m_-/m_+$, the ratio of electron mass to ion mass, is a small number. We expand in powers of ϵ in all equations keeping an appropriate scale length fixed. This length is $\sqrt{\ell_+\ell_-}$ where ℓ is the distance a particle moves forward in a complete change of velocity phase, that is, a gyro-period, (see Appendix 1).

One of the keys to finding a solution by an expansion procedure is to take $E_{x'}$, the charge separation field, of the form $E(x',\epsilon)/\epsilon$ where E is finite as $\epsilon \longrightarrow 0$. In other words, there is a large charge separation field due to the disparity in the masses of the ions and electrons. Another vital point is to note that for $\epsilon \neq 0$, no matter how small, all ions travel from $-\infty$ to $+\infty$ provided $H_{z'} > 0$. It will be shown that as $\epsilon \longrightarrow 0$ this requires that special care must be used in determining the asymptotic behavior of the ions, since there are no permanently trapped ions and the behavior to lowest order in ϵ seems at first glance to imply that there are. In computing, we have to make the assumption that there is a point $x' = x'_*$ such that for $x' > x'_*$ the charge separation field dominates and for $x' < x'_*$ the magnetic field dominates.

The results which are proved in the body of this report, for lowest order in ϵ, are as follows: There is a number $R'_0 < U$ such that

I. If $R' \leq R'_o$, all ions go from $-\infty$ to $+\infty$ without turning around. The magnetic field has a pulse but all quantities return at $+\infty$ to the state they were in at $-\infty$, see Figure 1. This solution confirms to lowest order in ϵ certain special results, the well-known limiting solution for zero pressure ($R' = 0$), see Section 4 and also [1], and Mittleman's unpublished results. There is no shock and the flow is completely reversible. Simple two-fluid theories (both adiabatic) without standard dissipative mechanisms lead to similar flows (not shocks), see [1], [7], [8], [12]. Refined two-fluid theories have so far also not led to shocks. According to Lüst's approximation, a two-fluid theory with non-symmetric pressures does not yield a shock solution, see also [15].

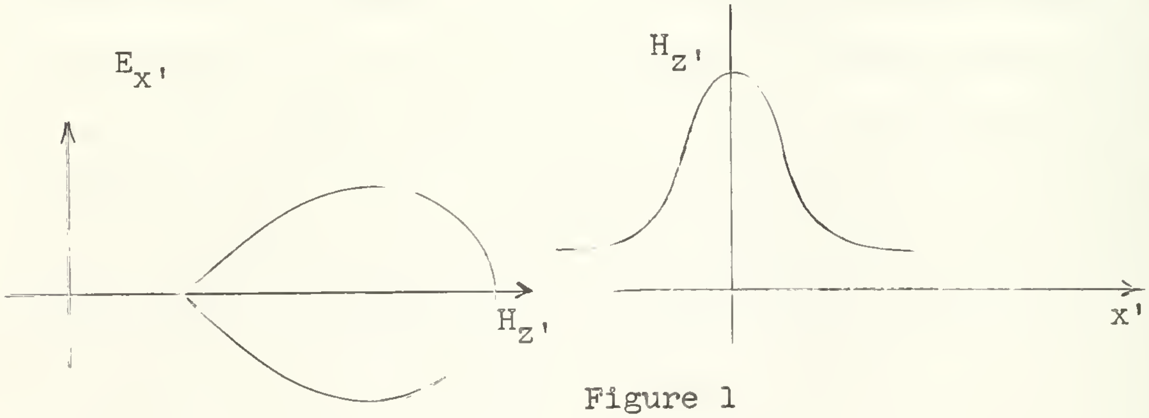

Figure 1

II. $R_o' < R' < U$. Some ions bounce a few times. In doing so they increase the density in front of the magnetic pulse. As a result the symmetry about, say $x' = 0$, disappears and the solution behaves for $x' < 0$ like the first part of the single pulse but for x' > 0 it oscillates without damping. See Figure 2.

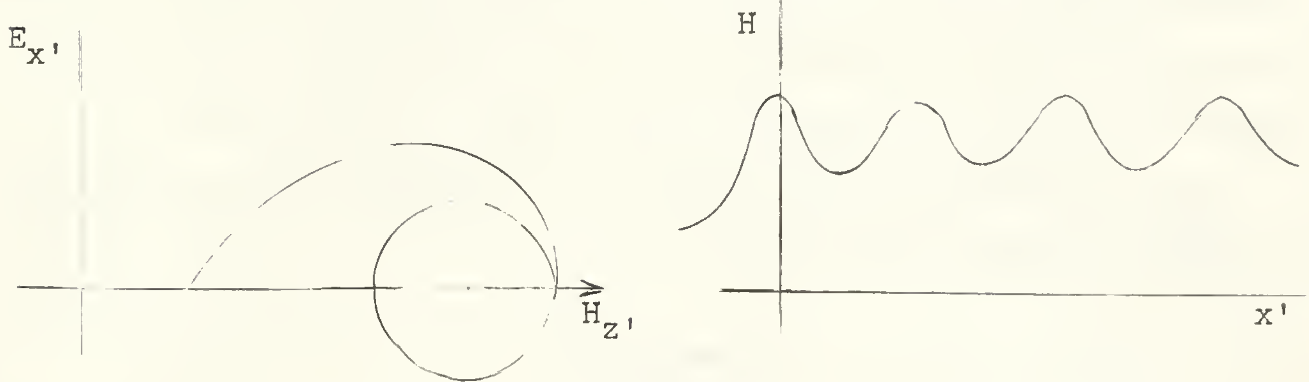

Figure 2.

2. Equations and Boundary Conditions.

The equations of our system are Maxwell's equations for a steady state electromagnetic field and two Boltzmann equations without collision terms. The former are

$$(1) \quad \begin{cases} \operatorname{curl} \vec{H} = \vec{J} \\ \operatorname{curl} \vec{E} = 0 \\ \operatorname{div} \mu\vec{H} = 0 \\ \operatorname{div} K'\vec{E} = q \end{cases}$$

where $\vec{H}$ is magnetic field, $\vec{E}$ is electric field, $\vec{J}$ is current, q is net charge per unit volume, K' and μ are vacuum constants[*].

The two Boltzmann equations reduce to

$$(2) \qquad m_{\pm}\, \vec{u} \cdot \nabla_{x'} f'_{\pm} \;\pm\; e(\vec{E} + \vec{u} \text{ x } \mu\vec{H}) \cdot \nabla_{u'} f'_{\pm} = 0$$

since there are no collisions. Here $m_{\pm}$ is the mass (of the ion or electron), $\vec{u} = (u',v',w')$ is the velocity vector. The space variables are x',y',z'; e is the electron charge and $f'_{\pm}(u',v',w',x')$ is the distribution function (for ions or electrons), i.e., the number of ions in an element of phase space per unit volume.

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[*] The units in the above equations are m.k.s.

The current and charge in Maxwell's equations (1) are related to the distribution functions $f'_{\pm}$ by

$$(3) \qquad \begin{cases} \vec{J} = e \iiint (f'_+ - f'_-)\vec{u}\, du'dv'dw' \\ \\ q = e \iiint (f'_+ - f'_-) du'dv'dw' \end{cases}$$

where the domain of integration is the whole space.

We assume that all quantities depend on one space variable only, say x'. As a result we see from (1) that $H_{x'}, E_{y'}, E_{z'}$ are all constant. We take $H_{x'} = E_{z'} = 0$. We prescribe $H_{y'} = 0$ at $-\infty$ and we look for a solution with $H_{y'} \equiv 0$. The two varying fields are $E_{x'}$ and $H_{z'}$ and we prescribe

$$(4) \qquad \begin{cases} E_{x'} = 0 \;, \quad H_{z'} = H^{\infty}_{z'} \\ \\ f'_{\pm} = F'_{\pm}(u',v',w') \qquad \text{for } x' = -\infty. \end{cases}$$

The prescribed distribution functions must be consistent with a constant state.

The problem of finding a strict shock solution is to find a solution of (1)-(4) that tends to a constant state as $x' \longrightarrow +\infty$ and which differs from the state at $-\infty$. No such solutions have been found. In fact no exact solutions to (1)-(4) except the solutions with zero pressure have been determined. Attempts to find solutions numerically by forward differences in x' have been unsuccessful because the characteristics of (2) in the u',v',w',x'-space may intersect the planes x' = constant more than once. The present method for studying solutions is to perform asymptotic expansions in a suitable parameter with appropriately normalized variables.

But first we introduce some notation and derive certain conditions for $x' \longrightarrow -\infty$ which are useful in normalizing the variables. We introduce the number densities

(4a) $$n_{\pm} = \iint f'_{\pm}\, du'dv'dw'$$

and the mean velocities

(4b)
$$\bar{u}'_{\pm} = \iint u' f'_{\pm}\, du'dv'dw' / \iint f'_{\pm}\, du'dv'dw'$$
$$\bar{v}'_{\pm} = \iint v' f'_{\pm}\, du'dv'dw' / \iint f'_{\pm}\, du'dv'dw' .$$

Integrating the Boltzmann equations (2) we find

(4c) $n_+\bar{u}'_+$ and $n_-\bar{u}'_-$ are constant.

Since the fields tend to constants as $x' \longrightarrow -\infty$ we see from (1) that $q \longrightarrow 0$ and $J \longrightarrow 0$. From (3)

$$J = e(n_+\bar{u}'_+ - n_-\bar{u}'_-) \text{ and } q = e(n_+ - n_-),$$

and hence $n_+ - n_- \longrightarrow 0$ as $x \longrightarrow \infty$, and $\bar{u}'_+ - \bar{u}'_- \longrightarrow 0$. Thus setting $n_{\pm} = n^{\infty}_{\pm}$ and $\bar{u}'_+ = U$ at $-\infty$, we have

(5) $$n_+^\infty = n_-^\infty \qquad \text{for } x = -\infty$$

(6) $$\bar{u}_+' = \bar{u}_-' = U \qquad \text{for } x = -\infty$$

and

(7) $$n_+\bar{u}_+' = n_-\bar{u}_-' = n_+^\infty U \qquad \text{for } x = -\infty.$$

Note that U, the mean value of the speed of the ions and the electrons as $x' \longrightarrow -\infty$, is the speed of the shock viewed from a frame of reference in which the mean speed of the gas is zero and the shock is moving.

From (2) we have as $x' \longrightarrow -\infty$, since $f_\pm'$ become independent of x',

$$(\vec{E} + \vec{u} \times \vec{H}) \cdot \nabla_{u'} f_\pm = 0$$

and hence using (4) and (6) we find that

(7a) $$E_{y'} = \mu U H_{z'}^\infty .$$

We now introduce an intetrated distribution function

(7b) $$f_\pm = \int_{-\infty}^{+\infty} f_\pm' \, dw' .$$

Then equation (2) holds for $f_\pm$ instead of $f_\pm'$. The next step is to normalize the variables.

We want to expand with respect to the square root of the mass ratio, and we find that the appropriate length to keep fixed is given by

(8) $$x = e\sqrt{\frac{n_+^\infty \mu}{m_-}} \; x' .$$

The other suitable dimensionless variables are

$$(9)\quad \begin{cases} u = u'/U \\ v = v'/U \\ H = H_{z'}/H_{z'}^{\infty} \\ E = \epsilon E_{x'}/E_{y'} \end{cases}$$

with

$$\epsilon = \sqrt{m_-/m_+} \quad .$$

Note that with this choice of variables, $E_{x'} \longrightarrow \infty$, as $\epsilon \rightarrow 0$.

Using (7) and (3), Maxwell's equations (1) reduce to

$$(10)\qquad \frac{dH}{dx} = -A\epsilon\left(\frac{\bar{v}_+}{\bar{u}_+} - \frac{\bar{v}_-}{\bar{u}_-}\right) \quad , \qquad K\frac{dE}{dx} = \left(\frac{1}{\bar{u}_+} - \frac{1}{\bar{u}_-}\right)$$

where* $K = \frac{K'}{A\epsilon^2}\frac{U^2}{c^2}$ and A is the Alfven-Mach number, $A = \sqrt{m_+ n_{+\infty} U^2/\mu H_{z'}^{\infty 2}}$, $c = 1/\sqrt{K'\mu}$ is the speed of light, $\bar{v}_\pm$ is the mean velocity in the y-direction after normalization, that is;

$$(11)\qquad \bar{v}_\pm = \iint f_\pm v\ du\ dv / \iint f_\pm du\ dv = n_\pm^{-1} \iint f_\pm v\ du\ dv,$$

and $\bar{u}_\pm$ is the normalized mean velocity in the x-direction, that is;

---

[*] We note that K' is about 10^{-7}. Hence if U/C is sufficiently small compared to ϵ^2 so that K is small compared to 1, we may take $\bar{u}_+ = \bar{u}_-$. In this case the total charge is zero. This approximation which is frequently made will be valid unless dE/dx turns out to be very large. However, it is not necessary to make this assumption, and we do not.

(12) $$\bar{u}_\pm = \iint f_\pm u \, du \, dv \Big/ \iint f_\pm du \, dv = n_\pm^{-1} \iint f_\pm u \, du \, dv.$$

For a suitable choice of the distribution functions at $-\infty$ we have $\bar{v}_\pm = 0$ at $-\infty$. On the other hand by (4a), (4b) and (4c), (6) and (7b),

(12a) $$\iint f_\pm u \, du \, dv = 1.$$

With the change of variables (7b), (8), (9), the two Boltzmann equations (2) reduce to

(13) $$\begin{cases} Au \dfrac{\partial f_+}{\partial x} + (E + \epsilon vH) \dfrac{\partial f_+}{\partial u} + \epsilon(1-uH) \dfrac{\partial f_+}{\partial v} = 0 \\[2ex] Au \dfrac{\partial f_-}{\partial x} - \dfrac{1}{\epsilon} \left(\dfrac{E}{\epsilon} + vH\right) \dfrac{\partial f_-}{\partial u} - \dfrac{1}{\epsilon} (1-uH) \dfrac{\partial f_-}{\partial v} = 0. \end{cases}$$

From the boundary conditions (4) at $-\infty$, we have with (9)

(14) $$H - 1 = E = 0 \,, \quad \text{at } x = -\infty;$$

(15_+) $$f_+ = f_+^\infty \,, \qquad \text{at } x = -\infty;$$

(15_-) $$f_- = f_-^\infty \,, \qquad \text{at } x = -\infty.$$

We restrict f_+^∞ to satisfy

(16) $$\begin{cases} f_+^\infty = F((1-u)^2 + v^2) \neq 0 & \text{for } (1-u)^2 + v^2 \leq R^2 \\[1ex] \quad = 0 & \text{for } (1-u)^2 + v^2 > R^2. \end{cases}$$

The cut-off parameter $R = R'/U$ will be specified later. We impose here only the condition

(16a) $$R < 1.$$

This means that for $x \to -\infty$ the ions all have positive velocity in the x-direction and a speed less than 2. The function f_-^∞ may be any monotone function of speed at $-\infty$.

The problem is to solve (10), (11), (12), (13) under the conditions (14), ($15_\pm$), (16) and (16a). We require that E,H are bounded and have bounded derivatives for all ϵ.

We have in addition from (11), (12), (6), (16)

$$\bar{u}_+ - 1 = \bar{u}_- - 1 = \bar{v}_+ = \bar{v}_- = 0 \quad \text{at } x = -\infty \tag{16b}$$

From (13) we see that the function f_- is constant along the electron particle paths:

$$\left\{ \begin{aligned} \frac{du}{dt} &= u\,\frac{du}{dx} = -\frac{1}{A\epsilon}\left(\frac{E}{\epsilon} + vH\right) \\ \frac{dv}{dt} &= u\,\frac{dv}{dx} = -\frac{1}{A\epsilon}(1 - uH) \end{aligned} \right. \tag{17}$$

where $dy = dx/u$ is the differential of the time parameter on a particle path.

On the other hand, the function f_+ is constant along the ion particle paths:

$$\left\{ \begin{aligned} \frac{du}{dt} &= u\,\frac{du}{dx} = \frac{1}{A}(E + \epsilon vH) \\ \frac{dv}{dt} &= u\,\frac{dv}{dx} = \frac{\epsilon}{A}(1 - uH). \end{aligned} \right. \tag{18}$$

The method for finding an approximate solution is the following. First we find asymptotic expansions in ϵ for $(11)_-$ and $(12)_-$ for the electrons. It is then possible to find integrals of equations (10) and (13) and we are left with a Boltzmann equation for the ions with coefficients which depend on moments of the ion distribution function, (Section 3). We then look separately at the various characteristics (18) for the ion Boltzmann equation and their dependence on x for $\epsilon \longrightarrow 0$, (Sections 5 and 7) and thereby draw conclusions about the behavior of E and H (Sections 6 and 8).

3. Asymptotic Behavior of the Mean Velocities of the Electrons and Resulting Momentum Integrals.

The second equation of (13) for the distribution function for the electrons will now be solved for E and H by an asymptotic series in ϵ, and from this expansion, the average values of $\overline{u}_-, \overline{v}_-$ which occur in (10) may be determined. We have then a simplified problem to solve.

However, it is not possible to find directly from (13) an expansion of f_- in a power series in ϵ. A preliminary transformation is necessary and this transformation is suggested by an approximation, very similar to the "guiding center" approximation but differing from it because of the large charge separation field.

We note that if we set $v = - E/\epsilon H$, $u = 1/H + \frac{d}{dx}\left(\frac{AE}{H}\right)$ in (17) the equations are satisfied to the highest order in ϵ. This suggests introducing as new independent variables

$$(19) \quad \begin{cases} \xi' = - \dfrac{1}{H + \dfrac{d}{dx}\left(\dfrac{AE}{H}\right)} + u \\[2em] \eta' = + E/\epsilon H + v. \end{cases}$$

With these new variables it is possible after some calculations, which are given in Appendix 2, to find an asymptotic expansion* for f_-. Substituting this expansion in (11) and (12), one finds to order ϵ^2

$$(20) \quad \begin{cases} \bar{u}_- = \dfrac{1}{H + \dfrac{d}{dx}\left(\dfrac{AE}{H}\right)} \\ \\ \epsilon \bar{v}_- = - E/H. \end{cases}$$

We might interpret this by saying that in the mean the electrons behave like a guiding center fluid. There exists an adiabatic invariant for this motion and it is therefore possible to show that the electrons are heated adiabatically. (This adiabatic invariant is not, however, the magnetic moment.)

Substituting (20) in (10) we find, along with (13), that the problem is reduced to solving

$$(21) \quad \begin{cases} \dfrac{d}{dx}\left(H + \dfrac{1}{2}\eta^2\right) = \eta H - \dfrac{A\epsilon v_+}{\bar{u}_+} \\ \\ \dfrac{d}{dx}\left(\eta + \dfrac{K\eta H}{A}\right) = - \dfrac{1}{\bar{u}_+} + H \\ \\ Au \dfrac{\partial f_+}{\partial x} + \left(- \dfrac{\eta}{A} + \epsilon v\right) H \dfrac{\partial f_+}{\partial u} + \epsilon (1 - uH) \dfrac{\partial f_+}{\partial v} = 0 \end{cases}$$

where $\bar{u}_+$ and $\bar{v}_+$ are given by (11) and (12) and $\eta = -AE/H$. The boundary conditions reduce to

$$(22) \quad \begin{cases} H - 1 = \eta = 0 \quad \text{at } x = -\infty \\ \\ f_+ = f_+^{\infty} \end{cases}$$

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* For a general theory of such asymptotic expansion see a forthcoming report by C. S. Gardner.

Fields $E = -\eta H/A$ and H which satisfy (21) and (22) will satisfy $(10\text{-}15_{\pm})$ to terms of order ϵ^2.

Energy and momentum integrals of the equations.

Two integrals of equations (21) can be found corresponding to the conservation of momentum flux in the x and y directions. Multiplying the last equation of (21) by u and integrating by parts over the (u,v)-space we find

$$A \frac{d}{dx} \iint f_+ u^2 du\, dv + \iint \left(\frac{\eta}{A} - \epsilon v\right) H f_+ du\, dv = 0$$

or using the first two equations of (21), with (11), (12) and (12a):

$$(23a) \qquad I(x) + \frac{H^2}{2A^2}(1 - K\eta^2) = \text{constant}$$

where

$$(23aa) \qquad I(x) = \iint f_+ u^2 du\, dv.$$

The constant is found using (22), to be $I(-\infty) + 1/2A^2$.

Similarly, one finds by multiplying the Boltzmann equation by v,

$$(23b) \qquad AJ(x) + \epsilon\eta\left(1 + \frac{KH}{A}\right) = 0$$

where $J(x) = \iint f\, uv\, du\, dv$.

It can be shown that (23aa) and (23b) may be solved for η and H in terms of I and J. The functions I and J are simply related to the stress tensor.

An additional integral relation involving the electric potential $\phi = -\int E dx$ proves useful in the lowest order case. From the first equation of (21) we note, by integrating from $-\infty$ to x, that

$$\text{(23c)} \qquad A\phi = -A\int_{-\infty}^{x} E\, dx = \int_{-\infty}^{x} \eta H\, dx = -1 + H + \frac{\eta^2}{2} + \epsilon A \int_{-\infty}^{x} \frac{\overline{v}_+}{\overline{u}_+}\, dx$$

$$= \text{constant} + H + \frac{\eta^2}{2} + \epsilon A \int_{x_*}^{x} \frac{\overline{v}_+}{\overline{u}_+}\, dx \quad \text{for any } x_*.$$

4. The Simplified Problem

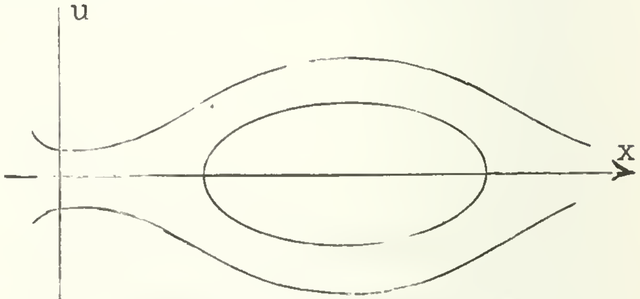
The original boundary value problem has been reduced to solving the Boltzmann equation for the ions, (the last equation of (21)):

$$\text{(24)} \qquad Au \frac{\partial f_+}{\partial x} + \left(-\frac{\eta}{A} + \epsilon v\right) H \frac{\partial f_+}{\partial u} + \epsilon(1-uH)\frac{\partial f_+}{\partial v} = 0$$

where η and H are given in terms of the moments $I(x) = \iint f_+ u^2 du\, dv$ and $J(x) = \iint f_+ uv\, du\, dv$ through (23a) and (23b) with one condition prescribed at $-\infty$, namely, $f_+ = f_+^\infty$ where f_+^∞ is given by (16). We also have (23c) which turns out to be easier to use than (23b).

We would like to know when there exists a solution to this problem aside from the trivial solution $f_+(u,v,x) = f_+^\infty(u,v)$. First of all one can show, (see Calkin [3] for the general proof) that there exists a solution near $x = -\infty$ where f_+ increases exponentially in x. Then by (23a) and (23b), η and H change exponentially also. We want to know if this solution can be continued in x.

In accordance with the general procedure so far we would like next to find an asymptotic expansion of (24), (23a), (23b), with the boundary condition (16), to lowest order in ϵ. However, we run into two difficulties if we look only at the lowest order terms. First of all, it turns out that we cannot solve to lowest order and satisfy the boundary condition at $-\infty$, (16), in a straightforward fashion. Secondly, as we shall see, to lowest order in ϵ, there are closed characteristics and since they plainly never reach $-\infty$ it seems that we could prescribe f_+ arbitrarily on them. In this case the solution does not appear to be unique.*



Lowest order characteristics in x,u plane.

However, these difficulties can be overcome by using the second order terms in the expansion and some general arguments about the behavior of the characteristics. The correct procedure is to use the lowest order term in the expansions in ϵ to decide on what curves f_+ is constant and to use more refined arguments to decide what value of f_+ to associate with these curves. The major part of the work is concerned with the latter problem. Once we know to lowest order in ϵ the curves on which f_+ is constant and qualitatively what the constant is, we can draw conclusions from (23a) and (23c) about the behavior of H and η.

In the next section (5) we shall discuss the very simple case of zero pressure which gives us some insight into some general properties of the solution of (24). Then in Section 6, we return to the problem of finding the lowest order characteristics and attaching values of f_+ to them.

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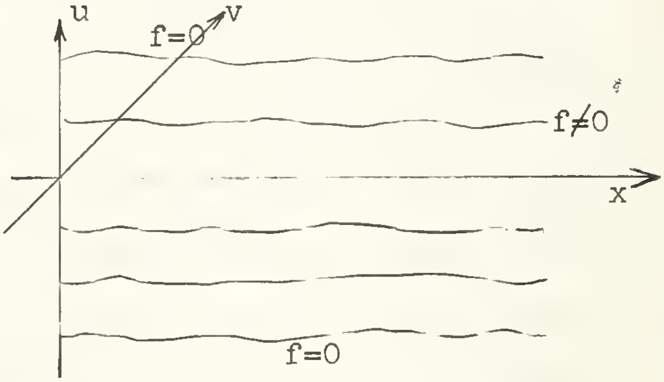
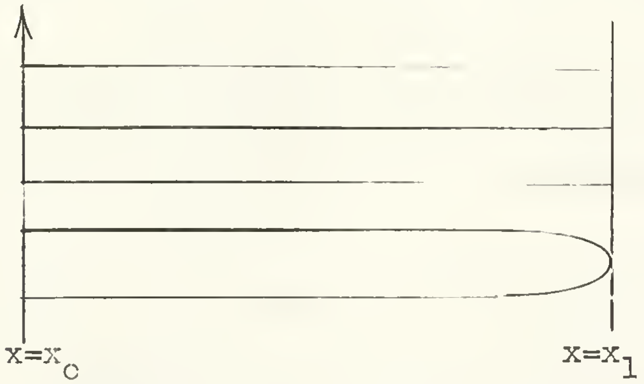
* See [2].

In the rest of this section, we shall draw some conclusions about characteristics in general. Now we consider general properties and characteristics. We use interchangeably the words "particle path" and "characteristic".

We recollect that f_+ is constant on the characteristics which are given by (18). Now, if for all $x > x_o$ we have $u > 0$ on the particle paths issuing from $x = x_o$, f_+ is determined for $x > x_o$ by its values on $x = x_o$. Now suppose that some particle paths loop, i.e., u changes sign at $x = x_1$, say, and a particle path comes back to $x = x_o$. Then, on this particle path a certain value of f is carried along, see figure, and plainly f_+ cannot be arbitrarily prescribed on $x = x_o$.





However, in our problem under certain simple and reasonable assumptions about the field it will be shown that no particle path will return to $x = x_o$ if x_o is sufficiently large and negative. Therefore, we conclude that f^+ is determined everywhere by its values for large negative x.

The necessary assumptions are that $|\emptyset| = |-\int E\,dx|$ is bounded and that H is bounded and bounded away from zero. These conditions hold for all approximate solutions that are found here. The above statement about the behavior of the ions is a special case of a more general lemma.

Lemma 1: Every characteristic of (24)(given by (18)) extends from $x = -\infty$ to $x = +\infty$.

This lemma is equivalent to the statement that there are no trapped ions and will be proved later.

A second lemma will be needed which tells us how v increases with time. A characteristic which loops and crosses some plane x = constant several times will have different values of u and v associated with each time it crosses such a plane.

Lemma 2: On a characteristic, the values of v corresponding to the same value of x increase with time.

Proof: Along any characteristic we may introduce the time t, where $dx/dt = u$, as a parameter and then x is a function of t. From (18), we have on a characteristic

$$\frac{u^2}{2} = \frac{u_*^{\,2}}{2} - \frac{\phi(x)-\phi(x_*)}{A} + \epsilon\int_{x_*}^{x} v\,H dx$$

(24a)

$$v = v_* + \frac{\epsilon(t-t_*)}{A} - \frac{\epsilon}{A}\int_{x_*}^{x} H dx$$

where the integrals are along the characteristic and $u = u_*$, $v = v_*$, $x = x_* \neq +\infty$ for $t = t_*$.

The proof of lemma 2 follows immediately from the second line.

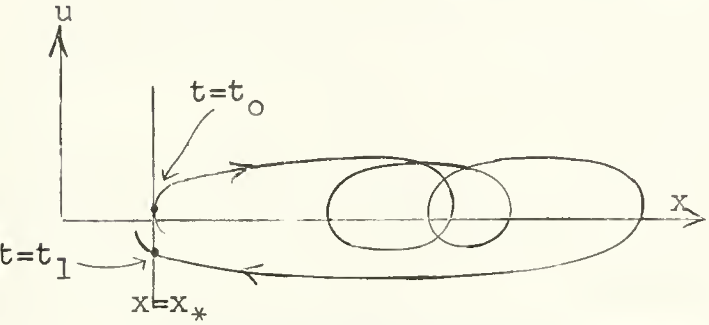
A third lemma is needed to prove the first.

Lemma 3: On a characteristic, the values of x where u changes from negative to positive (or positive to negative) increase with increasing time.

This statement is equivalent to the statement that an ion has moved forward each time it just turns around.

Proof of Lemma 3:

Suppose the lemma is not true. Then for say $x = x_*$ we have, at some time t_o, $u = u_* = 0$. Let u_1 be the value of u when the ion first comes back to $x = x_*$ at time $t = t_1$.



We have from the first line of (24a),

$$\frac{u_1^2}{2} = \epsilon \int v\ Hdx.$$

Consider the path between these two events, $t_o \leq t \leq t_1$, as consisting of forward (f) and backward (b) transits ($dx > 0$, $dx < 0$) of an ion.

Then $\epsilon \int v\ Hdx$ may be written as $\epsilon \int_{(f)} v\ Hdx - \epsilon \int_{(b)} vH|dx|$. Now at each value of x every forward transit is matched by a later backward transit. By lemma 2 the value of v is larger on the backward transit than the value of v on the corresponding forward transit. Hence the integral $\epsilon \int_{t_o}^{t_1} v\ Hdx$ is negative and therefore cannot equal $u_1^2/2$ which is positive. Therefore the ion does not return to $x = x_*$ and the first half of the lemma is true. The second half is proved in the same way.

Proof of Lemma 1: Suppose the lemma is not true. Then as $t \to \infty$, $x < x_r < \infty$, for some ion. From the second equation of (24a) for $t \to \infty$, $\int_{x_*}^{x} H\,dx \not\to \infty$ since $x < x_r$. Hence $v \to \infty$.

By lemma 3 there is some x_ℓ and t_ℓ such that for $t > t_\ell$, we have $x > x_\ell$. Suppose u vanishes for $x = x_* > x_\ell$ at time t_* taken so large that $v > \frac{3|E|_{max} + k}{A\epsilon H_{min}}$, $k > 0$, where $|E|_{max}$, $H_{min} \neq 0$ are the respective maximum and minimum on $x_\ell \leq x \leq x_r$. Then by (24a), on the first forward transit after t_*

$$A\frac{u^2}{2} > -2|\phi|_{max} + \int_{x_*}^{x} (3|E|_{max} + k)dx$$

$$> |\phi|_{max} + k_1(x - x_*)$$

where $|\phi|_{max} = \max\limits_{x_\ell < x < x_r} |\phi|$, k_1 is constant.

Hence $u \neq 0$ for $x > x_*$ and the ion cannot turn around again. Therefore, for sufficiently large times u does not change sign. Hence, by the second equation of (18), since $v \to \infty$, $dv/dt = u\,dv/dx = \epsilon(1-uH) \to \infty$, and thus $dv/dx = \epsilon(\frac{1}{u} - H) \to \infty$ since $u \neq 0$. Therefore, $u \to 0$ through positive values. But, by (18) $\frac{1}{2}du^2/dx = E + vH \to \infty$ and therefore $u^2 \not\to 0$. Thus the hypothesis cannot hold and the lemma is proved.

5. The Special Case of Zero Pressure.

The case in which f_+^∞ is a delta function* can be treated completely. The higher-than-first moments of f_+^∞ about the mean velocity vanish and therefore this case corresponds to zero pressure. We may also say that all the ions have the same velocity for all values of x. The mean velocities $\bar{u}_+$ and $\bar{v}_+$ are easily shown to satisfy (18) with $u = \bar{u}_+$, $v = \bar{v}_+$ provided u is positive along the particle path. Furthermore, from (23a) and (23b) we find explicitly using (16b) and (12a),

$$\begin{aligned} &\bar{u}_+ + \frac{H^2}{2A^2}(1-K\eta^2) = \frac{1}{2A^2} \\ &A v_+ + \epsilon\eta\left(1 + \frac{KH}{A}\right) = 0. \end{aligned} \tag{25}$$

From the first equation of (24a), and (22)

$$\bar{u}_+^2 - 1 = -\frac{2}{A}\phi + \frac{2\epsilon}{A}\int_{-\infty}^{x} \bar{v}_+ H\,dx. \tag{26}$$

These three equations yield with (23c) and (24),

$$\bar{u}_+^2 - 1 = \frac{2}{A^2}\left(1 - H - \frac{\eta^2}{2}\right) - \frac{\epsilon^2}{A^2}\eta^2\left(1 + \frac{KH}{A}\right)^2 . \tag{27}$$

The relation between η and H is derived by eliminating $\bar{u}_+$ between (25) and (27),

$$1 + \frac{2}{A^2}\left(1-H-\frac{\eta^2}{2}\right) = \left(-1 + \frac{1}{2A^2}(H^2-1-K\eta^2H^2)\right)^2 + \frac{\epsilon^2}{A^2}\eta^2\left(1+\frac{KH}{A}\right)^2. \tag{28}$$

---

* The following solution is essentially well-known and has been established by many authors, either using the particle viewpoint as here or using a two-fluid approach with zero pressure, see for example [1], [5], [7].

The solution curve in the (H,η)-plane is illustrated in Figure 1. It exists provided $\overline{u}_+^2 = 1 + \frac{2}{A^2}(1-H- \eta^2/2) + O(\epsilon^2) > 0$. It is plainly symmetric about $\eta = 0$ and hence the solution is a pulse, see Figure 2. There is no shock. It is also clear that qualitatively the terms in K have no effect.

We can find a maximum value, A_{max}, of A such that if $A < A_{max}$, $\overline{u}_+ > 0$ for all η,H satisfying (28). We may take $\epsilon = 0$ and $K = 0$ for convenience. From (25), for fixed A, $\overline{u}_+$ has its minimum where H has its maximum and by (28), this occurs where $\eta = 0$. Hence at the maximum value of A, $\overline{u}_+ = 0$ where $\eta = 0$. Therefore,

$$1 + \frac{2}{A^2}(1-H) = 0 \quad \text{and} \quad 1 + \frac{1}{2A^2}(1-H^2) = 0.$$

Hence $A_{max} = 2$.

We shall throughout restrict ourselves to the case in which $A < 2$. One can easily show that for x sufficiently large and negative, $E = -d\phi/dx < 0$ and $dH/dx > 0$. In the case in which f_+ at $-\infty$ is not a delta function but a cut-off Maxwellian, these two conditions will hold if the pressure is sufficiently small, i.e., if the distribution function f_+^∞ is concentrated in a narrow region. We assume this holds.

6. The General Motion of the Ions.

We proceed to find solutions of (18) as series in ϵ and thereby a formal power series solution of (24). These series we write as

$$(29) \quad \begin{cases} u = u_o + \epsilon u_1 + \ldots \\ v = v_o + \epsilon v_1 + \ldots \end{cases}$$

We assume E and $H > 0$ are given independently of ϵ, continuous and bounded for $-\infty \leq x \leq +\infty$. The latter condition can be verified after finding an approximate solution. We find, on equating coefficients of powers of ϵ in (18) and integrating,

$$(30) \quad \begin{cases} u_o^2 = u_*^2 - \frac{2}{A}(\emptyset - \emptyset_*) \\ v_o = v_* \\ u_1 u_o = \frac{1}{A} v_* \int_{x_*}^{x} H\, dx \\ v_1 = \frac{1}{A}\left\{\int_{x_*}^{x} dx/u_o - \int_{x_*}^{x} H dx\right\} \end{cases}$$

where $u_* = u(x_*)$, $v_* = v(x_*)$, $\emptyset_* = \emptyset(x_*)$ and integrals are taken along particle paths. We write $\int_{x_*}^{x} dx/u_o = t_o$, the time that the ion takes in the zero order approximation to go from x_* to x. The position x_* may be chosen arbitrarily to suit the particular application but we shall often choose it to correspond to a small value of $\emptyset$ and a large negative value of x, small compared with $1/\epsilon$.

The expression for v_1 is not necessarily a single-valued function of x. It assumes a different value each time the ion with initial velocity u_*, v_* crosses $x = x_*$.

The first order approximation in ϵ is, by (30),

$$u^2 \sim u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*)) + \frac{2\epsilon}{A} v_*[P(x) - P(x_*)]$$

(31)

$$v \sim v_* + \frac{\epsilon}{A}\left[\int_{x_*}^{x} \frac{dx}{\sqrt{u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*))}} - P(x) + P(x_*)\right]$$

where the square root has the same sign as dx and $P(x) - P(x_*) = \int_{x_*}^{x} H dx$. The neglected terms are $O(\epsilon^2)$. We must consider where (31) is valid.

We note immediately that the expansion to zero order in ϵ is not valid* for $|x-x_*|$ large of order $1/\epsilon$; for then, $\epsilon P(x)$ becomes $O(1)$.

Also $\int_{x_*}^{x} dx/\sqrt{u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*))}$ is not singular where $u_*^2 - \frac{2}{A}\phi + \frac{2}{A}\phi_*$ vanishes unless $E = -d\phi/dx$ also vanishes. Therefore (31) will be valid for all ions as long as $x-x_*$ is bounded and $u_*^2 - 2/A\,(\phi(x)-\phi(x_*))$ does not vanish at $E = 0$.

The exceptional situation where $E = 0$ and $u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*))$ vanishes can be treated separately by assuming E is linear in x near this point. It can then be shown by a lengthy but straightforward computation that the ion paths either satisfy (31) to lowest order or their corresponding contributions to the integrals in (23aa) (23c) tend to zero as $\epsilon \longrightarrow 0$.

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* We shall not therefore be able to say anything about the motion of the particles and hence about the fields for x positive and greater than $o(1/\epsilon)$. But the original problem is not correctly set if we look for a solution in the range $-\infty < x < X$ where $X < +\infty$ unless we prescribe on $x = X$ the falues of f_+ (and even of f_-) for negative values of u. We are therefore forced to impose some conditions for x large. These rather weak conditions will be discussed in case d) of this section.

Note that because of the condition on $x-x_*$ we are not justified in using the leading term alone at $-\infty$. This condition would not have appeared if we had not examined the first order terms in ϵ.

On the other hand we may apply (31) when $|x-x_*|$ is $O(1)$ and v_* is $o(1/\epsilon)$. This is established formally in the following way: We set $E^* = E + \epsilon v_* H$, $v = v_* + V$. Then by (18),

$$u \frac{du}{dx} = \frac{1}{A} (E^* + \epsilon VH)$$

$$u \frac{dV}{dx} = \frac{\epsilon}{A} (1 - uH)$$

with $u = u_*$, $V = 0$ for $x = x_*$. These equations are just like (18) and have bounded initial conditions. A power series development yields

$$u^2 = u_*^2 + \frac{2}{A} \int_{x_*}^{x} E^* dx + O(\epsilon^2)$$

$$V = \frac{\epsilon}{A} \int_{x_*}^{x} \frac{dx}{\sqrt{u_*^2 + \frac{2}{A} \int_{x_*}^{x} E^* dx}} - P(x) + P(x_*) + O(\epsilon^2).$$

But $\int_{x_*}^{x} E^* dx = \int_{x_*}^{x} (E + \epsilon v_* H) dx = - \emptyset(x) + \emptyset(x_*) + \epsilon v_* (P(x) - P(x_*))$ and thus we get, neglecting terms that are $o(\epsilon)$,

$$u^2 = u_*^2 - \frac{2}{A}(\emptyset(x)-\emptyset(x_*)) + \frac{2\epsilon}{A} v_*(P(x)-P(x_*))$$

$$v = v_* + V = v_* + \frac{\epsilon}{A}\int_{x_*}^{x} \frac{dx}{\sqrt{u_*^2 - \frac{2}{A}(\emptyset(x)-\emptyset(x_*))}} - T(x)+T(x_*)$$

as in (31).

We shall use (31) in two situations; a) where $|x-x_*|$ is $o(1/\epsilon)$ and v is $O(1)$; b) where $|x-x_*|$ is $O(1)$ and v is $o(1/\epsilon)$.

We find, in using (31) that there there are four kinds of ion paths to discuss: the "through" ion paths which do not turn around ($u > 0$) in any finite region; the "pseudo-trapped" ion paths which to lowest order in ϵ appear to be closed paths*; the "bouncing" ion paths which to lowest order appear to be reflected back to $-\infty$ at the potential peak, and "returning" ion paths which appear to come from $+\infty$. Our objective is to find out what value of f_+ should be associated with each of these paths. (See figure at the end of this section.)

There are also paths on which v is at least of order $1/\epsilon$. They are not described by (31). However, we are spared finding a complete description of these paths, as we shall see, by the fact that $f_+ = 0$ there.

a) "Through" ion paths. From (31) we see that to first order in ϵ, an ion turns back (i.e., u changes sign) only if

$$0 = u^2 = u_*^2 - \frac{2}{A}(\emptyset - \emptyset_*) + \frac{2\epsilon}{A} v_*(P(x)-P(x_*)).$$

To zero order in ϵ an ion follows a "through" ion path ($u > 0$)

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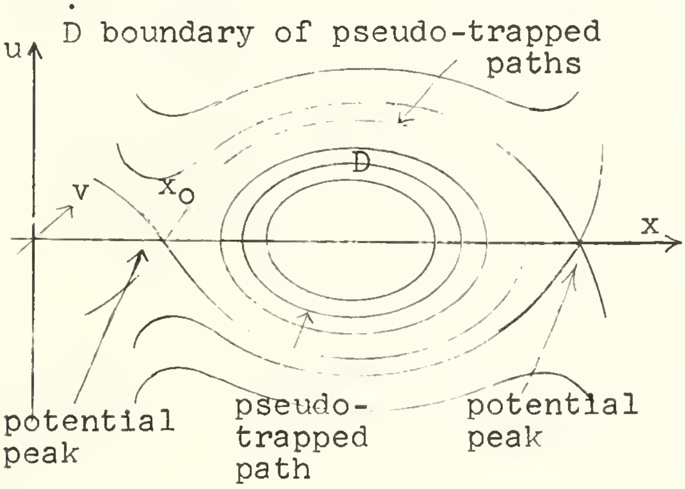
* We recollect from Lemma 1 that there are no trapped ions.

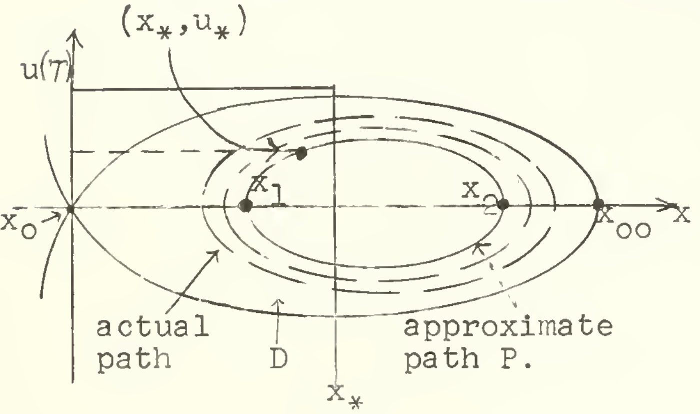
if its initial velocity u_* satisfies $u_*^2 - \frac{2}{A}(\phi_{max} - \phi(x_*)) > 0$ provided $v_* << 1/\epsilon$. For these ions we note the important fact that v also remains constant to lowest order in ϵ.

b) "Pseudo-trapped"ion paths. Let us consider the ion paths to zero order in ϵ in (u,x)-space with $v = o(1/\epsilon)$. Between two peaks of potential the paths appear as in the adjacent sketch. Take any path that is a closed curve — we call the corresponding ions pseudo-trapped ions since no ion really is trapped (Lemma 1). We want to show that for our problem, that is to first order in ϵ, there are no ions ($f_+ = 0$) on these paths.



We prove this in several parts. First we assume there is a pseudo-trapped particle, i.e., such a particle path with $f_+ \neq 0$. Then we find a condition on the y-component of velocity for this ion when it starts being trapped. From this we find similar conditions when it first enters the region where there is a sizable charge separation firld (E). This condition in turn implies conditions on the speed of the ion at $-\infty$. Finally we notice by the cut-off condition that there are no ions satisfying this condition at $-\infty$.

Note that the ion we are concerned with is characterized by the conditions $x = x_*$, $u = u_*$, $v = v_*$ for $t = t_*$. These conditions are independent of ϵ. Its earlier position and velocities are in general functions of t and of ϵ.



Suppose an ion is near a point on one of these curves at some time with $v = O(1)$. That is, let $u = u_*$, $v = v_* = O(1)$, $x = x_*$ for $t = t_*$ and let x_1, x_2 be such that $u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*))$ vanishes for $x = x_1$ and $x = x_2$. Note that the curve

$$u^2 = u_*^2 - \frac{2}{A}(\phi(x)-\phi(x_*))$$

lies entirely inside D, see figure.

We want to determine where the corresponding ion came from and where it is going to. We assert Statement 1: This ion came from a region with v at least of the order $1/\epsilon$.

The proof is based on the following idea: in the (x,u,v)-space, the characteristic actually winds around a cylinder whose cross-section is the approximate path P where $v = o(1/\epsilon)$. It differs from P only for large v. A particle moving on this path is in this region for a long time. We will show that we arrive at a contradiction if we assume v is $o(1/\epsilon)$ on the whole path between x_o and x_{oo}.

Proof: Suppose $v = o(1/\epsilon)$ for the ion, while its path lies between x_o, x_{oo} up to the time t_* when $u = u_*$, $v = v_*$. Then on any cycle we may apply (31). If, at any time t_1 prior to t_*, this ion has a value of x_1, u_1 lying in D, its path for $t_2 < t < t_1$ lies arbitrarily close (if ϵ is made sufficiently small) to the pseudo-trapped path passing through x_1, u_1. Here t_2 is the preceding value of t when the ion crossed x_1 with positive u (i.e., one complete cycle). Therefore, to enter D the ion must cross the boundary $\dot{D}$ of D, and on its first cycle in D its path lies arbitrarily close to $\dot{D}$. In other words, on its first cycle, by (31), letting $\epsilon \longrightarrow 0$,

$$u^2(x) \sim u_{**}^2 - \frac{2}{A} (\phi(x) - \phi(x_*))$$

where $u_{**} = u_{**}(\epsilon)$ is the value of u on $\dot{D}$ for $x = x_*$. Let $T(\epsilon)$ be the corresponding value of t with $x = x_*$ and $u(T) \sim u_{**}$. By (24a), we have exactly

$$u^2(T) = u_*^2 + 2\epsilon \int_{t_*}^{T} v \; H \; dx$$

where the integral is taken (backwards in time) along the path from $t = t_*$ to $t = T$. Hence, combining these two relations,

$$u_{**}^2 \sim u_*^2 + 2\epsilon \int_{t_*}^{T} v \; H \; dx.$$

Since (u_*, x_*) lies inside D and (u_{**}, x_*) lies on the boundary $\dot{D}$, we have $u_*^2 < u_{**}^2$. Thus u_{**} and u_* differ by a quantity that is O(1) and does not go to zero as $\epsilon \longrightarrow 0$. Therefore $\epsilon \int_{t_*}^{T} v \; H dx$ is positive and bounded away from zero as $\epsilon \longrightarrow 0$. But

$$\epsilon \int_{t_*}^{T} v\, H dx = \epsilon \int_{t_*}^{T} v\, dP = \epsilon[vP]_{t_*}^{T} - \epsilon \int_{t_*}^{T} P\, dv.$$

Since v is $o(1/\epsilon)$, and $\epsilon[v]_{t_*}^{T} \sim o(1)$, we conclude that $\epsilon \int Pdv$ is negative and bounded away from zero as $\epsilon \longrightarrow 0$. But $\epsilon \int Pdv = \epsilon \int P \frac{dv}{dx} dx = \epsilon^2 \int P(\frac{1}{u} - H)dx$ by (18) or $\epsilon \int Pdv = \epsilon^2 \int Pdt - \epsilon^2 \int P\, dP = \epsilon^2 \int Pdt - \frac{\epsilon^2}{2} [P^2]_{t_*}^{T}$ since $dx = udt$. The last term is zero since $P = P(x)$ and $x = x_*$ for $t = t_*$ and $t = T$. Therefore we have $\epsilon \int Pdv = \epsilon^2 \int Pdt$. But $|\epsilon^2 \int Pdt| < \epsilon^2 |P|_{max} |T-t_*|$. Hence since we have already shown that $\epsilon \int Pdv$ is bounded away from zero as $\epsilon \longrightarrow 0$, it follows that $\epsilon^2 |T-t_*|$ is bounded away from zero. By (24a), $v(T) = v_* + \epsilon(T-t_*) - \epsilon[P]_{t_*}^{T} = v_* + \epsilon(T-t_*)$, and we conclude that $\epsilon(v(T) - v_*)$ is bounded away from zero as $\epsilon \longrightarrow 0$. Since $\epsilon v_* \longrightarrow 0$ as $\epsilon \longrightarrow 0$, by assumption, we conclude that, as $\epsilon \longrightarrow 0$, $\epsilon v(T)$ is bounded away from zero or $v(T) \neq o(1/\epsilon)$ which contradicts our assumption. Statement I on p. 33 is proved.

Furthermore, we shall now show Statement II: this ion first crossed any plane $x = x_*$ with $|x_* - x| = o(1/\epsilon)$ at a time $t = t_*$ with v negative if $v \neq o(1/\epsilon)$. We also derive an inequality to use later.

Let $\tilde{v}$, $\tilde{t}$ be the values of v, t when the ion first crosses $x = \tilde{x}$. We recall that v_*, t_* are the corresponding values when the ion crosses $x = x_*$. Then on integrating the second equation of motion we have, with $dx = udt$,

$$v_* - \tilde{v} = \epsilon \int_{\tilde{t}}^{t_*} (1-uH)dt = \epsilon[t_* - \tilde{t}] - \epsilon \int_{\tilde{x}}^{x_*} Hdx$$

or

$$\lim_{\epsilon \to 0} \epsilon(v_* - \tilde{v}) = \lim_{\epsilon \to 0} \epsilon^2 [t_* - \tilde{t}] - \lim_{\epsilon \to 0} \epsilon^2 \int_{\tilde{x}}^{x_*} H dx.$$

Now H is bounded, and $\lim\limits_{\epsilon > 0} \epsilon(x_* - \tilde{x}) = 0$ by assumption. Hence

$$\lim \epsilon\tilde{v} = \lim \epsilon v_* - \lim \epsilon^2 (t_* - \tilde{t}) = \lim \epsilon^2 (\tilde{t} - t_*).$$

Also for any two times $\tilde{t}_1, \tilde{t}_2$ with $v = \tilde{v}_1, \tilde{v}_2$, $x = \tilde{x}_1, \tilde{x}_2$,

$$(*) \qquad \lim \epsilon(\tilde{v}_1 - \tilde{v}_2) = \lim \epsilon^2 (\tilde{t}_1 - \tilde{t}_2).$$

Since $t_* > \tilde{t}$, we see from the first inequality

$$\lim_{\epsilon \to 0} \epsilon\tilde{v} < 0,$$

and Statement II is proved.

By Statement I, at some time t_1, with $x = \tilde{x}_1$ such that $x_o \leq \tilde{x}_1 \leq x_{oo}$ and $v = \tilde{v}_1$, $\lim \epsilon\tilde{v}_1 \neq 0$. By Statement II, $\lim \epsilon\tilde{v}_1 \leq 0$. Hence $\lim \epsilon\tilde{v}_1 < 0$. Combining this result with equation (*), we see that for $t = \tilde{t}_2 < \tilde{t}_1$ we have

$$(**) \qquad \lim \epsilon\tilde{v}_2 = \lim \epsilon\tilde{v}_1 - \lim \epsilon^2 (\tilde{t}_1 - \tilde{t}_2) < 0$$

at $t = \tilde{t}_2 < \tilde{t}_1 < t_*$, $x = \tilde{x}_2 < x_*$ where $|\tilde{x}_2 - x_*| = o(\frac{1}{\epsilon})$ and $v = \tilde{v}_2$.

We recall that this inequality refers to an ion which is pseudo-trapped near $x = x_*$. We will next prove a third statement about such an ion and show that this statement contradicts the prescribed condition (16a) at $-\infty$. We pick $\tilde{x}$ sufficiently far to the left so that for all ϵ, $E \leq -a$, say, for $x \leq \tilde{x}$. [In the region to the left we cannot use the approximation (31) since $|E|$ is small but it will be valid with appropriate restrictions on v to the right.]

The next step is to estimate v in $x < x_*$. We set $E = \epsilon\mathcal{E}$, $X = x/\epsilon$.

We may assume, since it is true for the zero pressure case, that $\mathcal{E} < 0$ and $d\mathcal{E}/dx < 0$. Then $\mathcal{E}$ changes monotonically from 0 to $-a/\epsilon$.

The equations of motion become

$$(***) \qquad u\,\frac{du}{dx} = \mathcal{E} + vH, \qquad u\,\frac{dv}{dx} = 1 - uH.$$

Because the parameter ϵ no longer appears in the equations of motion, there is no simple expansion procedure for estimating v. However, it is reasonable to treat H as constant in this range and it is then possible to make various estimates for the velocities. These estimates are given in Appendix 3. We have then, see Lemma A, with $r > 0$ being the value of $r = \sqrt{(1-u)^2 + v^2}$ at $-\infty$,

$$v > -r$$

at each point on the ion path* in the region where $\mathcal{E} < 0$ and $\frac{d\mathcal{E}}{dx} < 0$. In particular this holds for the first transit of the plane $x = x_*$. By (***), it follows that $\lim\limits_{\epsilon \to 0} \epsilon r > 0$. We have proved

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* Note that the ion in question is characterized by the condition that it crosses $x = x_*$ with $u = u_*$, and $v = v_*$ where x_*, u_*, v_* are independent of ϵ. Everything else depends on ϵ. Hence in particular, r depends on ϵ.

Statement III: in the limit as $\epsilon \longrightarrow 0$, an ion which is at some time pseudo-trapped near $x = x_*$ has at $-\infty$ a value of r which becomes infinite like $1/\epsilon$. But, by (16a), there are no particles with $r > R$, i.e., $f_+^\infty(u,v) = 0$ there. (Note R is fixed independent of ϵ.) Hence there are no particles on the pseudo-trapped paths, i.e., $f_+(u,v,x) = 0$ there.

We have established, therefore, that there are no pseudo-trapped ions, i.e., no ions on the closed paths given by

$$u^2 + \frac{2\phi}{A} = 2\,\frac{\phi(x_1)}{A} = 2\,\frac{\phi(x_2)}{A}\ , \quad v = \text{constant}$$

with $x_1 \neq x_2$. And thus $f_+ = 0$ there.

c) "Bouncing ion paths". These are ion paths which to lowest order in ϵ are of the form

$$u^2 + \frac{2\phi}{A}(x) = \frac{2\phi}{A}(x_1) \quad , \quad v = \text{constant}$$

but along which $\phi(x) \neq \phi(x_1)$ for $x \neq x_1$.

For this class of ion paths u changes sign only if $x = x_1$ is in the region where (31) is valid, that is, $x > x_*$ say. It follows that the corresponding ion moves with increasing x from x_* to x_1 and back to x_*. For $x < x_*$ the magnetic field turns the ion around and it recrosses $x = x_*$ for some new value of u_*. At this stage the ion follows a path which may or may not be of the above form. If it is, with say $u^2 + (2/A)\phi(x) = (2/A)\phi(x_2)$ the ion is again turned around at $x = x_2$ and the process is repeated. According to Lemma 2 this ion acquires an increment of v each time it turns around. Eventually it passes through (Lemma 1). In this final stage we must have

$$u^2 + \frac{2}{A} \phi > \frac{2}{A} \phi_{max}$$

where ϕ_{max} is the maximum value of ϕ.

On the one hand, going backwards in time on the path of this ion we see that, since the ion bounces, it must cross $x = x_*$ initially with a positive value u less than $\sqrt{(2/A)\phi_{max}}$.

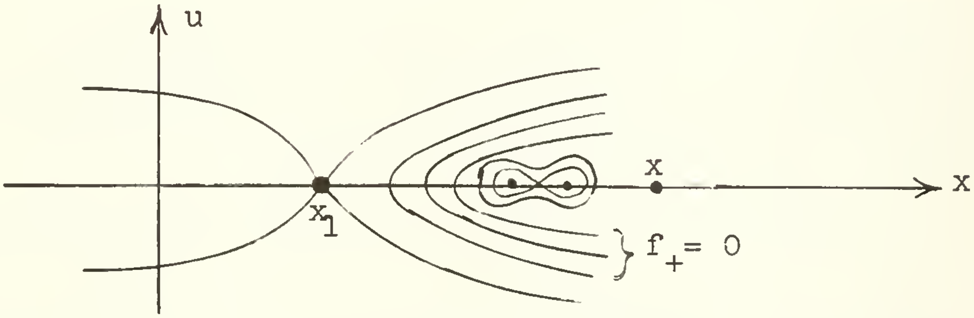
On the other hand, for $x < x_*$ it can be reasoned that u does not change on the average and that some particles with $(1-u)^2 + v^2 = r^2$ at $-\infty$ will have $u < 1-r$ for $x = x_*$. The argument is given in Appendix 3 where the necessary estimates are developed. Hence, if $u = \sqrt{(2/A)\phi_{max}}$ cuts $(1-u)^2 + v^2 = r^2 < R^2$ there will be some bouncing ions, see (16a). Of course, these bouncing ions are going to influence the field for $x < x_*$ but if there are not too many of them and if they do not bounce too often, they do not change the general character of the field for $x < x_*$. This effect will be discussed in section 10.

However, their effect for $x > x_*$ is of the greatest significance as we shall see.

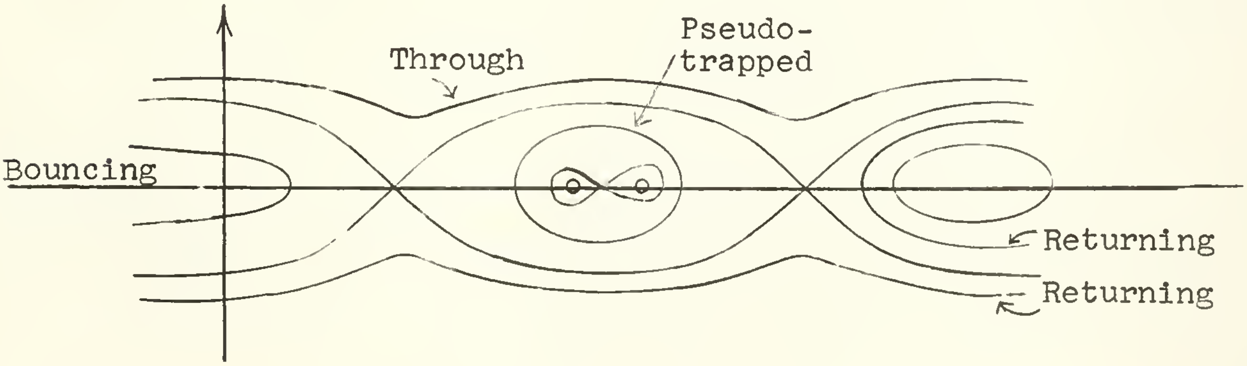
It is now possible to describe the behavior of all ions which, at $x = -\infty$, have velocities lying in the circle $(1-u)^2 + v^2 \leq R^2$ where by (16a), $R < 1$. They either pass through with v unchanged or they bounce, possibly at several places, somewhere between $-\infty$ and the point x_1 where ϕ assumes its maximum. It is true that if they bounce often enough, v may increase a great deal. We assume* that v remains O(1).

d) "Returning" ions. These are ions which appear to come from $+\infty$ in the lowest order approximation. By Lemma 1 there are no ions actually coming back from $+\infty$. They therefore must correspond from our preceding analysis to ions that (possibly after "bouncing") went "through".

In this case we assume (3) there are no returning ions for which u changes sign between x_1 and $x_1 + o(1/\epsilon)$. (This can be shown to be true if E = 0, H = constant for sufficiently large x, see figure.



The adjoining sketch indicates the four kinds of ion paths in the (x,u)-plane.



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* If the cut-off radius R is sufficiently small, no ions bounce. Hence by a continuity argument one finds that by increasing R one gets first ions that bounce only a few times. For these, v remains O(1).

Conclusions about ion paths.

We may now finally decide what path any ion follows from $-\infty$. An ion may pass through the whole region to large positive x without turning around. It if turns around it bounces back to large negative x but eventually passes through. On its whole path v is $o(1/\epsilon)$. An ion never gets on a pseudo-trapped path.

7. Estimating $\epsilon\bar{v}_+$ and Determining the Potential ϕ.

Now that we know the behavior of the ions we can apply our knowledge to find how the fields behave.

On through ion paths, v is approximately constant by (31) and hence v is $O(1)$. On bouncing ion paths, v does not change by an order of magnitude. There are no pseudo-trapped ions and by the assumption for returning ions v is $o(1/\epsilon)$. Hence by (11), $\bar{v}_+ = \iint f_+ v \, du \, dv \, / \iint f_+ du \, dv$, we see that $\epsilon\bar{v}_+ \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

Therefore, from (23c),

$$A\phi = \text{constant} + H + \eta^2/2. \tag{32}$$

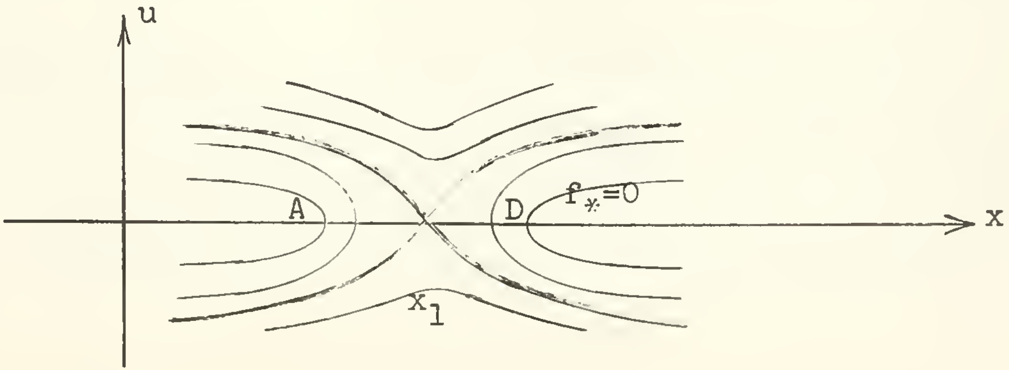
We also note that $\lim_{\epsilon\to 0} J = 0$ and $\lim_{\epsilon\to 0} \epsilon\eta = 0$. Hence (23b) yields no information in the lowest order case.

8. Boltzmann Distribution for Ions.

Having discussed in Section 6 the general motion of the ions we are now able to determine f_+ to zero order in ϵ for $-\infty < x < o(1/\epsilon)$. Then in the next section, we shall discuss the fields. By (31), $u^2 + \frac{2}{A}\phi = \text{constant}$ on ion paths where $\epsilon v_* \quad 0$. Hence $f_+(x,u,v)$ is constant on $u^2 + \frac{2}{A}\phi(x) = \text{constant}$. Hence f_+ is even in u for those values of u and x for which $u^2 + \frac{2}{A}\phi(x) < \frac{2}{A}\phi(x_1)$. On $x = x_*$, f_+ is determined on all particle paths, which meet $x = x_*$ for the first time, by its value at $-\infty$ on the corresponding particle path. Because of bouncing and returning, certain relations among the values of f_+ for some positive u hold, which are given roughly by taking $H = 1$, $\eta = 0$ for $x < x_*$, if x_* is small enough, but we do not need to know exactly what they are. We write for $x > x_*$

$$\begin{aligned} f_+(x,u,v) = f_1^+(u,v,\phi) &= F_1\left(u^2 + \frac{2\phi}{A}, v\right) \text{ for } u > 0 \\ &\qquad \text{or } u^2 + \frac{2\phi}{A} \leq \frac{2}{A}\phi(x_1) \\ &= G_1\left(u^2 + \frac{2\phi}{A}, v\right) \text{ for } u < 0 \text{ and } u^2 + \frac{2\phi}{A} > \frac{2}{A}\phi(x_1) \end{aligned} \tag{32a}$$

and these expressions will be valid as long as ϕ is monotonically increasing. Where ϕ reaches its first maximum at $x = x_1$ say, the ion paths $u^2 + \frac{2}{A}\phi = \text{constant}$ appear as in the figure. We will



show below that $f_+ = 0$ in the region D to the right. In the region A to the left f_+ is determined by the value of f_+ at $x = x_*$ and may or may not be zero.

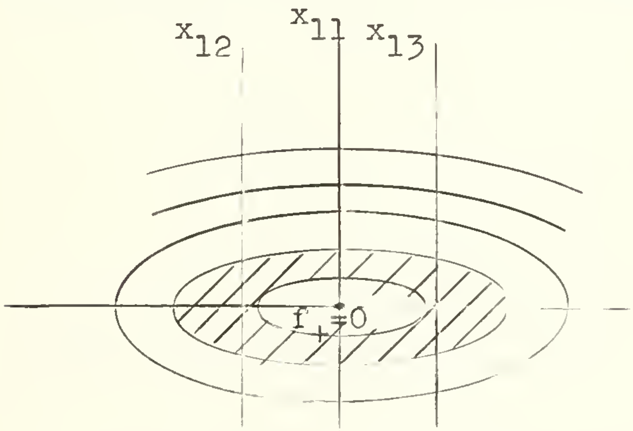
To show $f_+ = 0$ in D consider the two possible cases: either ϕ has its first minimum for $x > x_1$ at $x = X$ for

(1) X larger than $o(1/\epsilon)$ or

(2) $X = x_{11} = o(1/\epsilon)$.

In the first case we have the situation for which the third assumption about returning ions (case d) was designed. There are no ions in D. Hence $f_+ = 0$ in D.

In the second case let x_{12} and x_{13} be two values of x straddling x_{11} and such that $\phi(x_{12}) = \phi(x_{13})$ with $E = - d\phi/dx$ positive for $x_{12} < x < x_{11}$ and negative for $x_{11} < x \leq x_{13}$. Then $f_+(x_{12},u,v) = f_+(x_{13},u,v)$ since $\phi(x_{12}) = \phi(x_{13})$. From (23aa) and (23a) we see that I and hence $H^2(I-K\eta^2)$ have the same value at x_{12} and at x_{13}.



Also since $\phi(x_{12}) = \phi(x_{13})$, by ((32), p. 41), $H + \eta^2/2$ has the same value at the two points. Hence H and η^2 are the same for all such pairs of points. It follows furthermore from $\phi(x)-\phi(x_{11}) = -\int_{x}^{x} E dx = \int \frac{1}{A} H dx$ that $|x_{12} - x_{11}| = |x_{13} - x_{11}|$. Hence the curves in the x,u-plane on which $u^2 + 2\phi/A$ is constant are symmetric about x_{11} and therefore *D*, which is bounded by the curves $u^2 + 2\phi/A = \frac{2}{A}\phi_{max}$ *is closed at both ends since it is closed at one*. Therefore, inside D the ions are "pseudo-trapped" and hence $f_+ = 0$ in D.

We are now finally able to associate with each characteristic to lowest order in ϵ a value of f_+. If the characteristic comes from $x = x_*$, x_* large and negative, the value of f_+ on the characteristic is the value at $x = x_*$, and now we have shown that if it does not cut $x = x_*$, $f_+ = 0$.

9. Behavior of the Fields.

We now proceed to establish certain inequalities through which we will be able to draw qualitative conclusions about the fields.

We consider I as a function of x, see (23aa) and (32a),

$$
(33) \qquad \begin{cases} I = \iint f_+ u^2 du\, dv \\ \quad = \iint f_1^+ (u,v,\phi) u^2 du\, dv \\ \quad = I_1(\phi) & \text{for } x_* \leq x \leq x_1 \end{cases}
$$

where x_1 is the first maximum of ϕ, see (32a). Similarly

$$
(34) \qquad \begin{cases} I = \iint f_+ u^2 du\, dv = \iint f_2^+ (u,v,\phi) u^2 du\, dv \\ \phi = I_2(\phi) & \text{for } x_1 \leq x \leq x_2 \end{cases}
$$

where x_2 is the second maximum of ϕ. We note that $f_1(u,v,\phi) \geq f_2(u,v,\phi)$ since they are either equal or $f_2 = 0$. For through ions, $f_1 = f_2$. Hence $I_1(\phi) \geq I_2(\phi)$. In particular also $f_1 = f_2$ for $\phi = \phi_1$ and thus $I_1(\phi_1) = I_2(\phi_1)$. It follows, for $x > x_1$, that we must distinguish between two cases according to the value of the cut-off radius R, see (16a).

Case i) There are no bouncing ions and

$$I_1(\phi) = I_2(\phi).$$

Case ii) There are bouncing ions and

$$I_1(\phi) > I_2(\phi).$$

We can now discuss the fields.

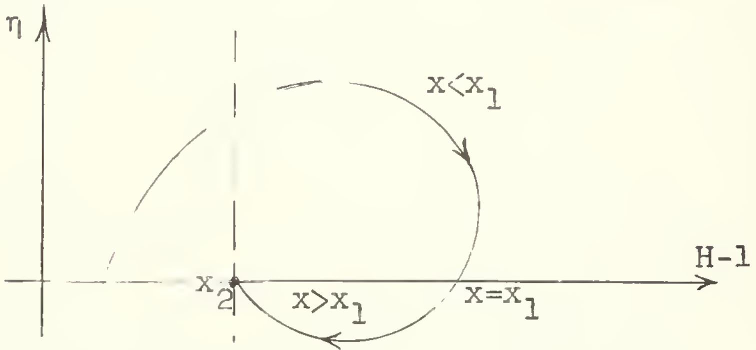
In case i) we see from (23a), (23aa), and (32) that each pair of values of x, one greater and one less than x_1 with the same value of ϕ also have the same value of H and η^2 and hence, by (32) and $d\phi = -Edx = \eta H\, dx/A$, the same value of $|x - x_1|$. Hence the solution is symmetric about $x = x_1$ and is a pulse. In the (η,H)-plane the solution is similar to that given in Figure 1 (Section 1). There is no shock. This situation can be achieved by taking the cut-off radius R small enough (see (16a)) since the solution will be as close as we like to the delta-function solution discussed in Section 4.

Case ii). In this case we see, using (32), (23a) and (23aa) that at two values of x straddling x_1, say $x_{21} < x_1$ and $x_{31} > x_1$, corresponding to the same value of H, we have

$$I_1(\phi(x_{21})) - KH^2\phi(x_{21})/A = I_2(\phi(x_{31})) - KH^2\phi(x_{31})/A.$$

Since along a path u^2 decreases with ϕ increasing, see (31),

$I_1(\phi)$ and $I_2(\phi)$ are decreasing functions of ϕ. The same is true of $I_1(\phi) - KH^2\phi/A$ and $I_2(\phi) - KH^2\phi/A$ for fixed H. Hence since $I_1(\phi) > I_2(\phi)$, $\phi(x_{21}) > \phi(x_{31})$. In turn, by (32), $\eta^2(x_{21}) > \eta^2(x_{31})$. Therefore, in the H,$\eta$-plane the solution curve, for $x > x_1$, lies inside the reflection of the solution for $x < x_1$ about $\eta = 0$, see figure.



The arrow indicates increasing x.

In particular, we may find a value of H - 1 so that the corresponding η is sufficiently small with $x < x_1$, and η for $x > x_1$ vanishes. At this point, $x = x_2$, ϕ has a minimum. By symmetry and arguments similar to those of Section 8, it can be easily shown that for $x > x_1$, $I = I_2(\phi)$ and hence the solution remains symmetric about $\eta = 0$ for $x > x_1$.

Character of fields.

If we look at the whole field over $x_* < x < +\infty$ we see that the magnetic field H increases and reaches a maximum value. It then oscillates between this maximum and some minimum that is greater than 1.

It is planned to find from computations with distributions that are cut-off Maxwellians at $-\infty$ the solutions for various cut-off radii R, gas-dynamic Mach numbers M at $-\infty$. and Alfven-Mach numbers A. A simple estimate shows that the minimum value of $u_*^2 - 2\emptyset/A$ with $f_+ \neq 0$ decreases linearly. Hence case ii) will always occur for fixed A and M if R is increased.

In these computations it is impossible to handle the charge separation field for $x < x_*$. Therefore, we simply take H = constant and E = 0 (or $\eta = 0$) for $x < x_*$. The lowest order theory for $x > x_*$ corresponds to taking H = 0 for $x > x_*$ for the ions.

A third case is perhaps worth mentioning although it is incomplete.

Case iii) With the above additional approximation for large negative x we find that an ion which arrives with u = 0, v > 0 at $x = x_*$ remains there. It cannot move forward, see (31) with $\epsilon = 0$, since $\sqrt{-2\emptyset/A}$ is imaginary. It cannot move backwards into the constant H field since $u^2 = \int \epsilon vHdx$ becomes negative. Hence the approximation cannot be valid. However, this bad approximation is still illuminating. We choose a cutoff radius so that no particle is stuck at $x = x_*$ in this approximation but turns around a great number of times before excaping. This means it acquires a very large value of v. Therefore, the distribution function for $x > x_*$ has a hump where v is very large. In other words, large distortions of the Maxwellian for large velocities are produced by a number of ions which started with small velocities. However, we have found numerically that if we let $R \rightarrow 1$, not only does the distortion become infinite, but also the pressure and current, so that the conditions for using a lowest order theory are violated. We, therefore, need a special theory for these ions and this has not been worked out as yet. Meanwhile the method should be applied only to cases where the ions do not turn around too often.

However, the singularities in the pressure and current indicate that some of the ions which at $-\infty$ have $R \simeq 1$ play in the full problem, i.e., where no approximations are made for $x < x_*$, a special role which should be analyzed but is probably qualitatively as described.

10. Actual Behavior of Fields as $x \longrightarrow -\infty$.

Certain properties of the fields for large negative x have been assumed. We would like to sketch here an asymptotic development for this region which seems to be consistent with our present knowledge.

We want to combine two limiting processes, letting $x \longrightarrow -\infty$ and $\epsilon \longrightarrow 0$, and we must do this in the right order. If we let $\epsilon \longrightarrow 0$ we see that the point at $-\infty$ corresponds to $1/\bar{u}_+ = H$ where $\bar{u}_+ = (\int\int f \, du \, dv)^{-1}$, see (12) and (12a). If we have bouncing ions and evaluate $\bar{u}_+$ to lowest order in ϵ for $x_* \longrightarrow -\infty$ we find that $\bar{u}_+(-\infty) < 1$ and hence $H(-\infty) > 1$, which contradicts the prescribed conditions. The correct process for finding the behavior at $-\infty$ is to let $x \longrightarrow -\infty$ and then $\epsilon \longrightarrow 0$. We may regard our lowest order solution as being a good approximation only as long as $|-x| << 1/\epsilon$ and $E >> \epsilon$, see (31). This situation is completely analagous to the usual boundary layer theory.

One cannot say very much about the behavior for $|-x| \sim 1/\epsilon$, but some qualitative remarks and justifications may be made. We.

return to equation (21) and for convenience let us set $K = 0$. We stretch certain of the variables. We set $\eta = \epsilon Y$, and $\epsilon x = X$. Then we have

$$\frac{dH}{dX} = Y(H-\epsilon^2 \frac{dY}{dX}) - \frac{A\overline{v}_+}{\overline{u}_+}$$

$$0 = \frac{1}{\overline{u}_+} - H + \epsilon^2 \frac{dY}{dX}$$

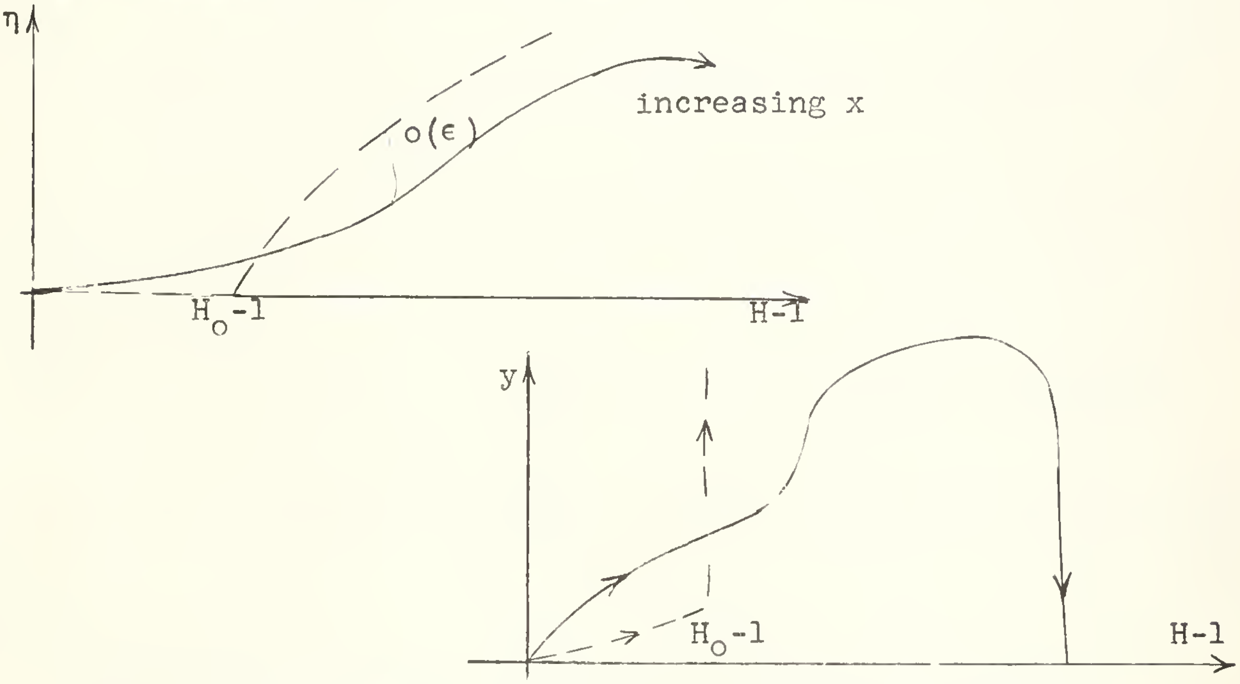
$$Au \frac{\partial f_+}{\partial X} + (-\frac{Y}{A} + v)H \frac{\partial f_+}{\partial u} + (1 - uH) \frac{\partial f_+}{\partial v} = 0.$$

Next we let $\epsilon \longrightarrow 0$ and find

$$(35) \quad \begin{cases} \frac{1}{H}\frac{dH}{dX} = Y + A\overline{v}_+ \\ \overline{u}_+ = \frac{1}{H} \\ Au \frac{\partial f_+}{\partial X} + (-\frac{Y}{A} + v)H \frac{\partial f_+}{\partial u} + (1 - uH) \frac{\partial f_+}{\partial v} = 0. \end{cases}$$

Now if no ions turn around $\overline{v}_+$ is small (it is zero at $-\infty$ and small at $x = x_*$) and can be neglected. This is confirmed in the zero pressure example where $\overline{v}_+ = O(\epsilon^2)$. But if there are ions coming back into the region where (35) is valid, from higher values of x, $\overline{v}_+$ will be positive since, 1) by Lemma 2, v increases at x with time and, 2) the contribution to $\overline{v}_+$ from the ions on their original passage to this value of x is small.

Let us suppose we have a zero order solution as described in Section 9 in which $H \longrightarrow H_o$ as $x \longrightarrow -\infty$. In the η,$(H-1)$-plane and Y,(H-1)-plane we have for $\epsilon \neq 0$ the solid curves, while if we let $\epsilon \longrightarrow 0$ we will obtain the dotted curves which look quite different in the two figures.



This behavior checks qualitatively with both the zero order solution and (35). Suppose we are approaching $H = H_o$ on the solid curve, $\epsilon \neq 0$, with decreasing x. We are within ϵ of the dotted curve in the $(\eta, H-1)$-plane and $\frac{1}{H}\frac{dH}{dx} \longrightarrow 0$ until η is $O(\epsilon)$. Then we must use (35) and we see that Y is not going to zero like dH/dx but remains larger because $A\bar{v}_+ > 0$. On the other hand, the density is decreasing because gradually we reach a region where no more ions are turning around. Therefore, $1/\bar{u}$ is decreasing, or, by (35), H is decreasing. Eventually, we reach a region which all the ions pass only once, the magnetic field is nearly constant, and the charge separation field is almost zero. The fields then approach their constant state at $-\infty$ in accordance with the behavior described by Calkin [3].

APPENDIX 1 - Unit of Length

We shall relate the unit of length introduced in (8) with the distances the ions and electrons move forward in a single period.

The equations of motion of an ion are, in m.k.s. units,

$$m_+ \frac{d^2x'}{dt'^2} = m_+ \frac{du'}{dt} = e(E_{x'} - \mu v' H_{z'})$$

$$m_+ \frac{d^2y'}{dt'^2} = m_+ \frac{dv'}{dt} = e(E_{y'} + \mu u' H_{z'}).$$

With $E_{x'} = 0$, $E_{y'}$ and $H_{z'}$ constant (and evaluated at $-\infty$), we find

$$u' - E_{y'}/\mu H_{z'} = (u'_o - E_{y'}/\mu H_{z'}) \cos(\frac{e}{m_+} \mu H_{z'} t) - v'_o \sin(\frac{e}{m_+} \mu H_{z'} t)$$

$$v' = (u'_o - E_{y'}/\mu H_{z'}) \sin(\frac{e}{m_+} \mu H_{z'} t) + v'_o \cos(\frac{e}{m_+} \mu H_{z'} t)$$

where u'_o, v'_o are the values of velocity at $t = 0$. The time T_+ for the ions to pass through a complete period is $2\pi m_+/e\mu H_{z'}$. The distance ℓ_+ an ion moves in T_+ is

$$\int_o^{T_+} u' dt = \frac{2\pi m_+}{e} \frac{E_{y'}}{\mu^2 H^2_{z'}} .$$

Similarly the distance ℓ_- an electron moves forward in $T_- = (2\pi m_-/e)(E_{y'}/\mu H^2_{z'})$ is

$$\ell = \sqrt{\ell_+ \ell_-} = \frac{2\pi}{\mu^3} \frac{E_{y'}}{\mu H_{z'}^2} \sqrt{m_+ m_-} = \frac{2\pi \sqrt{m_-}}{\sqrt{\mu n_{+\infty}}} \frac{E_{y'}}{\mu H_{z'}} \frac{\sqrt{m_{+\infty} n_{+\infty}}}{\sqrt{\mu H_{z'}}}$$

$$= 2\pi \frac{x'}{x} \frac{\sqrt{\rho_{+\infty}}\, U}{\sqrt{\mu}\, H_{z'}} = 2\pi A \frac{x'}{x}$$

by (8) and (9). Since in all cases considered $1 < A < 2$ we see that in the dimensionless system (8), (9) length is measured in units of ℓ. This is then the characteristic length of the motion and the width of the transition zones or the periods of the oscillation in case ii) will be some multiple of ℓ and if there are enough bouncing ions it will be of the same order as ℓ.

APPENDIX 2 - Asymptotic Behavior of Electrons

To determine asymptotic expressions for the mean electron velocities we transform the Boltzmann equation for the electrons using variables given in (18):

$$\xi' = u - \frac{1}{H - \frac{d\eta}{dx}} = u - \bar{u}$$

$$\eta' = v - \frac{1}{A\epsilon}\,\eta$$

$$f_-(x,u,v) = F_-(x,\xi',\eta').$$

Then the equation for F_- is:

$$A(\bar{u} + \xi')\,\frac{\partial F_-}{\partial x} - [A(\bar{u} + \xi')\frac{d\bar{u}}{dx} + \frac{\eta' H}{\epsilon}]\,\frac{\partial F_-}{\partial u'} + \frac{\xi}{\epsilon\bar{u}}\,\frac{\partial F_-}{\partial \eta'} = 0.$$

Setting $P = H^{1/4}\,\bar{u}^{-1/4}$, $Q = \bar{u}^{-3/4}\,H^{-1/4}$

$$\xi' = P(x)\Omega \cos\theta$$

$$\eta' = Q(x)\Omega \sin\theta$$

$$F_-(x,\xi',n') = F(x,\Omega,\theta)$$

we find with the notation $\bar{u}\,\frac{\partial\tau}{\partial x} = \dot{\tau}$, $\frac{\partial\xi}{\partial x} = \tau_x$

$$\bar{u}\,\frac{\partial F_-}{\partial x} = \dot{\tilde{F}} - \frac{\Omega}{2}\,(\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q})\,\frac{\partial\tilde{F}}{\partial\Omega} - \frac{\Omega}{2}\,(\frac{\dot{P}}{P} - \frac{\dot{Q}}{Q})\,\frac{\partial\tilde{F}}{\partial\Omega}\cos 2\theta$$

$$+ \frac{1}{2}\,(\frac{\dot{P}}{P} - \frac{\dot{Q}}{Q})\,\frac{\partial\tilde{F}}{\partial\theta}\sin 2\theta$$

$$\xi \frac{\partial F_{-}}{\partial x} = [P\Omega F_x - P\Omega^2(\frac{3}{4}\frac{P_x}{P} + \frac{1}{4}\frac{Q_x}{Q}) \frac{\partial F}{\partial \Omega}] \cos\theta$$

$$+ \frac{P\Omega}{4} (\frac{P_x}{P} - \frac{Q_x}{Q}) \frac{\partial \tilde{F}}{\partial Q} \sin\theta$$

$$- \frac{P\Omega^2}{4} (\frac{P_x}{P} - \frac{Q_x}{Q}) \frac{\partial \tilde{F}}{\partial \Omega} \cos 3\theta$$

$$+ \frac{P\Omega}{4} (\frac{P_x}{P} - \frac{Q_x}{Q}) \frac{\partial \tilde{F}}{\partial Q} \sin 3\theta ,$$

$$\frac{\partial F_{-}}{\partial \xi} = \frac{\cos\theta}{P}\frac{\partial \tilde{F}}{\partial \Omega} - \frac{\sin\theta}{P\Omega}\frac{\partial \tilde{F}}{\partial \theta}$$

$$- \bar{u}\,\frac{d\bar{u}}{dx}\frac{\partial F_{-}}{\partial \xi} = -(\frac{\dot{\bar{u}}}{P}\frac{\partial \tilde{F}}{\partial \Omega}) \cos\theta + (\frac{\dot{\bar{u}}}{P\Omega}\frac{\partial F}{\partial \theta}) \sin\theta$$

$$- \xi\,\frac{d\bar{u}}{dx}\frac{\partial F'}{\partial \xi} = \frac{\Omega}{2} (\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q}) \frac{\partial \tilde{F}}{\partial \Omega} + \frac{\Omega}{2} (\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q}) \frac{\partial \tilde{F}}{\partial \Omega} \cos 2\theta$$

$$- \frac{1}{2} (\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q}) \frac{\partial \tilde{F}}{\partial \theta} \sin 2\theta$$

$$- \frac{1}{\epsilon} (\eta H \frac{\partial F_{-}}{\partial \xi} - \frac{\xi}{\bar{u}}\frac{\partial F_{-}}{\partial \eta}) = \frac{1}{\epsilon} P^2 \frac{\partial F}{\partial \theta}$$

since $\frac{P}{\bar{u}Q} = P^2 = \frac{HQ}{P}$.

Adding all the above expressions together and noting that $d\bar{u}/dx = -(\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q})$ we find the differential equation for $F(x,\Omega,\theta)$ is:

$$\epsilon \left\{ \dot{\tilde{F}} + P\Omega \frac{\partial \tilde{F}}{\partial x} \cos\theta - [P\Omega^2(\frac{3}{4}\frac{P_x}{P} + \frac{1}{4}\frac{Q_x}{Q}) + \frac{\dot{\bar{u}}}{P}] \frac{\partial \tilde{F}}{\partial \Omega} \cos\theta \right.$$

$$+ [\frac{P\Omega}{4} (\frac{P_x}{P} - \frac{Q_x}{Q}) + \frac{\dot{\bar{u}}}{P\Omega}] \frac{\partial \tilde{F}}{\partial \theta} \sin\theta + \frac{\Omega\dot{Q}}{Q}\frac{\partial \tilde{F}}{\partial \Omega} \cos 2\theta -$$

$$- \frac{\dot{Q}}{Q} \frac{\partial \tilde{F}}{\partial \theta} \sin 2\theta - \frac{P\Omega^2}{4} \left(\frac{P_x}{P} - \frac{Q_x}{Q}\right) \frac{\partial \tilde{F}}{\partial \Omega} \cos 3\theta$$

$$+ \frac{P\Omega}{4} \left(\frac{P_x}{P} - \frac{Q_x}{Q}\right) \frac{\partial \tilde{F}}{\partial \theta} \sin 3\theta \Big\} + P^2 \frac{\partial \tilde{F}}{\partial \theta} = 0.$$

We may now find an asymptotic expansion for $\tilde{F} = \tilde{F}_o + \epsilon \tilde{F}_1 + \ldots$ by substitution and integration with respect to θ:

$$\tilde{F}_o = \mathcal{F}(x, \Omega)$$

$$\tilde{F}_1 = \frac{1}{p^2} \dot{\mathcal{F}}(x,\Omega)\theta + P\Omega \frac{\partial \mathcal{F}}{\partial x} \sin \theta - [P\Omega^2 \left(\frac{3}{4} \frac{P_x}{P} + \frac{1}{4} \frac{Q_x}{Q}\right)$$

$$+ \frac{\dot{\bar{u}}}{P}] \frac{\partial \mathcal{F}}{\partial \Omega} \sin \theta + \frac{\Omega \dot{Q}}{2Q} \frac{\partial \mathcal{F}}{\partial \Omega} \sin 2\theta$$

$$- \frac{P\Omega^2}{12} \left(\frac{P_x}{P} - \frac{Q_x}{Q}\right) \frac{\partial \mathcal{F}}{\partial \Omega} \sin 3\theta + \mathcal{F}_1(x, \Omega).$$

Since $\tilde{F}_1$ must be a single valued function of θ, $\dot{\mathcal{F}} = u \frac{\partial \mathcal{F}}{\partial x} = 0$. We may choose $\mathcal{F}_1 = 0$. The mean velocities are found using (11), (12) and $du\, dv = du'dv' = PQ\Omega d\Omega d\theta$,

$$u\, du\, dv = PQ\Omega(\bar{u} + P\Omega \cos \theta) d\Omega d\theta$$

$$v\, du\, dv = PQ\Omega(Q\Omega \sin \theta + \frac{\eta}{A\epsilon}) d\Omega d\theta.$$

We find

$$\bar{v}_- = \frac{\eta}{A\epsilon} + O(\epsilon) = - \frac{E}{H} + O(\epsilon)$$

$$\bar{u}_- = \bar{u} + O(\epsilon^2) = \frac{1}{H + \frac{d}{dx}\left(\frac{AE}{H}\right)} + O(\epsilon^2)$$

as required in (20).

APPENDIX 3 - Estimates

In this appendix we shall get some bounds on the velocities of an ion in a constant magnetic field but a decreasing charge separation field. In equations* (18) we take (A1) $H = 1$, $E \leq 0$, $\frac{dE}{dx} < 0$, $E(-\infty) = 0$. Equations (18) become

$$\text{(A1)} \qquad \begin{aligned} \frac{du}{dt} &= u\,\frac{du}{dx} = \frac{\epsilon}{A}\left(\frac{E}{\epsilon} + v\right) \\ \frac{dv}{dt} &= u\,\frac{dv}{dx} = \frac{\epsilon}{A}\,(1-u). \end{aligned}$$

These equations we regard as equations of motion for a particle path in u,v,x-space with t as time. Then from (18) we find

$$\text{(A2)} \qquad \frac{d}{dx}\left[(u-1)^2 + \left(v + \frac{E}{\epsilon}\right)^2\right] = 2\left(v + \frac{E}{\epsilon}\right) \cdot \frac{1}{\epsilon}\,\frac{dE}{dx}$$

or integrating along an ion path to a point P from some point P_∞ with $x = -\infty$ we have, using A1,

$$\text{(A3)} \qquad (u-1)^2 + \left(v + \frac{E}{\epsilon}\right)^2 = r^2 + 2\int_{P_\infty}^{P}\left(v + \frac{E}{\epsilon}\right)\frac{1}{\epsilon}\,\frac{dE}{dx}\,dx$$

where

$$\text{(A4)} \qquad r^2 = \left[(u-1)^2 + v^2\right]_\infty^P \quad .$$

The integral can be estimated first for certain P, namely those which correspond to a first transit. We say P belongs to a first transit if for $t < t(P)$, $x < x(P)$. The result is

---

* In general it is sufficient to take H constant, dE/dx of fixed sign.

Lemma A1: If P is a first transit, $v(P) > -r$.

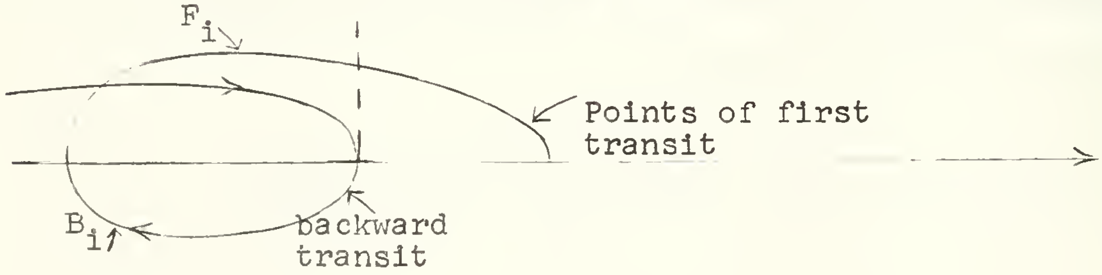
Proof: $\int_{P_\infty}^{P} (v + \frac{E}{\epsilon}) \frac{dE}{dx}\, dx$ may be written as the sums of the separate integrals along the forward transits and backward transits of the ions, i.e.,

$$\int_{P_\infty}^{P} (v + \frac{E}{\epsilon})\frac{dE}{dx}\, dx = \sum_{\text{forward}} \int (v + \frac{E}{\epsilon})\frac{dE}{dx}\, dx + \sum_{\text{backward}} \int (v + \frac{E}{\epsilon})\frac{dE}{dx}\, dx$$

(A4)

$$= \sum_{\text{forward}} \int (v + \frac{E}{\epsilon})\frac{dE}{dx} |dx| - \sum_{\text{backward}} \int (v + \frac{E}{\epsilon})\frac{dE}{dx} |dx|.$$

To each backward transit there corresponds an immediately subsequent forward transit. The first part of this forward transit has points which correspond to points on the backward transit and this is followed (see lemma 3) by points which are



of first transit, see figure. Thus the integral can be rewritten as

$$\int_{P_\infty}^{P} (v + \frac{E}{\epsilon})\frac{dE}{dx}\, dx = \sum_{i} \left[\int_{F_i} (v + \frac{E}{\epsilon})\frac{dE}{dx} |dx| - \int_{B_i} (v + \frac{E}{\epsilon})\frac{dE}{dx} |dx| \right]$$

(A5)

$$+ \int_{S(P)} (v + \frac{E}{\epsilon})\frac{dE}{dx} |dx|$$

where $S(P)$ is the set of points of first transit prior to P.

B_i is the set of points of the i^{th} backward transit and F_i is the set of points of the immediately subsequent forward transit that are not in S. Now,

$$\int_{F_i} (v + \frac{E}{\epsilon})\frac{dE}{dx}\,|dx| - \int_{B_i} (v + \frac{E}{\epsilon})\frac{dE}{dx}\,|dx| < 0$$

because the left hand side is, in obvious notation,

$$\int (v_{F_i} - v_{B_i})\,\frac{dE}{dx}\,|dx| \text{ over the x interval determined by } B_i.$$

But by (A1), $\frac{dE}{dx} < 0$, and by lemma 2, Section 4, $v_{F_i} > v_{B_i}$. Hence the integral is negative. Therefore

$$\text{(A6)} \qquad \int_{P_\infty}^{P} (v + \frac{E}{\epsilon})\frac{dE}{dx}\,dx < \int_S (v + \frac{E}{\epsilon})\frac{dE}{dx}\,|dx| < \int_S v\,\frac{dE}{dx}\,|dx| + \frac{E^2(P)}{2\epsilon}$$

by (A1). Substituting (A6) in (A3) we have

$$\text{(A7)} \qquad \left.(u-1)^2 + (v + \frac{E}{\epsilon})^2\right]_P < r^2 + \frac{2}{\epsilon}\int_S v\,\frac{dE}{dx}\,|dx| + \frac{E^2}{\epsilon^2}$$

or

$$\left.(v + \frac{E}{\epsilon})^2\right]_P < r^2 + \frac{2}{\epsilon}\int_{S(P)} v\,\frac{dE}{dx}\,|dx| + \frac{E^2}{\epsilon^2}.$$

Let V be the minimum value of v on S and let it be attained at $P = P_*$. Then since $E < 0$, $\frac{dE}{dx} < 0$, see (A1),

$$(V + \frac{E}{\epsilon})^2 < r^2 + \frac{2}{\epsilon}\int_{A(P_*)} v\,\frac{dE}{dx}\,|dx| + \frac{E^2}{\epsilon^2} < r^2 + \frac{2}{\epsilon}VE + \frac{E^2}{\epsilon^2}$$

or

(A8) $$v^2 < r^2$$

and the lemma is proved.

The second estimate concerns estimates for points that are not points of first transit.

Lemma A2: For all points

$$(u-1)^2+(v+\frac{E}{\epsilon})^2 \le (r+\frac{|E|_P}{E})^2 + \frac{2}{\epsilon}(r+\frac{2|E|_{\xi(P)}}{E})(|E|_{\xi(P)}-|E|_P),$$

where $\xi(P)$ is the maximum value of x on the characteristic up to P.

In (A5) we note that the end point of the integration may be in F_i or B_i. We find by similar arguments to those used above,

(A9) $$\int_{P_\infty}^{P}(v+\frac{E}{\epsilon})\frac{dE}{dx}\,dx \le \int_{S(P)}(v+\frac{E}{\epsilon})\frac{dE}{dx}\,|dx| - \int_{L}(v+\frac{E}{\epsilon})\frac{dE}{dx}\,|dx|$$

where L is that part of the last backward transit before P which is not covered by a subsequent forward transit. Now, by (A8),(A1),

(A10) $$\int_{S}(v+\frac{E}{\epsilon})\frac{dE}{dx}\,|dx| \le r|E| + \frac{E^2}{2\epsilon}\Big]_{x=\xi(P)}$$

where ξ is the maximum value of x on the path of the ion up to P. On the other hand, by lemma 3, the value of x at the right extreme point of L will be $\xi(P)$ and at the left extreme will be $x = x(P)$. Hence

(A11) $$-\int_{L}(v+\frac{E}{\epsilon})\frac{dE}{dx}\,|dx| = -\int_{L}v\frac{dE}{dx}\,|dx| + \frac{E^2}{2\epsilon}\Big]_{P} - \frac{E^2}{2\epsilon}\Big]_{\xi} .$$

Thus combining (A9),(A10),(A11), we have

(A12) $$\int_{P_\infty}^{P} (v + \frac{E}{\epsilon})\frac{dE}{dx}\, dx \le r|E| + \frac{E^2}{2\epsilon} \Big]_P - \int_L v \frac{dE}{dx}\, |dx| .$$

Substituting (A12) into (A3) we find

(A13) $$(v + \frac{E}{\epsilon})^2 \le (u-1)^2 + (v + \frac{E}{\epsilon})^2 \le (r + \frac{|E|}{\epsilon})^2 - \frac{2}{\epsilon}\int_L v \frac{dE}{dx}\, |dx| .$$

Let $v < V$ on L and let $v = V$ for some point P_L. Then, by (A13),

$$V^2 \le r^2 + \frac{2r|E|}{\epsilon} + \frac{2}{\epsilon}|E|_\xi V \le r^2 + \frac{2r|E|_\xi}{\epsilon} + \frac{2}{\epsilon}|E|_\xi \cdot V .$$

Hence

$$\left| V - \frac{|E|_\xi}{\epsilon} \right|^a \le (r + \frac{|E|_\xi}{\epsilon})$$

or

(A14) $$V \le r + \frac{2|E|_\xi}{\epsilon} \quad \text{on L.}$$

Substituting (A14) in the second inequality of (A13) we find for all points

(A15) $$(u-1)^2 + (v + \frac{E}{\epsilon})^2 \le (r + \frac{|E|_P}{\epsilon})^2 + \frac{2}{\epsilon}(r + \frac{2|E|_\xi}{\epsilon})(|E|_{\xi(P)} - |E|_P)$$

and the lemma is proved.

2. Behavior of ions that do not turn around.

Here we wish to give reasons why, when an ion passes through the region where E is small, the value of u does not change by an order of magnitude. We are concerned here with ions that are not turning around in this region. (They might turn around where E gets larger.)

The inequalities of Lemma A2 do not help. We need exact information about the trajectories which is impossible to get in general. We therefore consider the case in which E changes linearly from 0 to -a, say, over a very large distance so that we may consider $\ell = |dE/\epsilon\, dx|$ small compared with 1. Again H is taken to be unity without loss of generality. We set $E = -a - \epsilon \ell x$ over $-a/\ell \leq x \leq 0$, where $\ell > 0$. Then the equations of motion are, by (18),

$$\frac{du}{dt} = u\,\frac{du}{dx} = -a - \epsilon \ell x + \epsilon v$$

$$\frac{dv}{dt} = u\,\frac{dv}{dx} = \epsilon(1-u)$$

where $dx/dt = u$.

Differentiating with respect to t we have

$$\frac{d^2u}{dt^2} = -\,\epsilon \ell u + \epsilon(1-u)$$

and thus we find

$$u - \frac{1}{1+\ell} = \left[u - \frac{1}{1+\ell}\,\right]_{-a/\ell} \cos \epsilon\sqrt{1+\ell}\,\left(t - t\left(\frac{-a}{\ell}\right)\right)$$

$$+\ [v]_{-a/\ell} \sin \epsilon\sqrt{1+\ell}\,(t - t(-a/\ell)).$$

Setting $u = 1 + r\cos\xi$, $v = r\sin\xi$ at $-a/\ell$ we see that

$$\text{(A15)} \qquad u = 1 + \frac{\ell}{1+\ell}\,[\cos \epsilon\sqrt{1+\ell}\,(t - t(-a/\ell)) - 1]$$

$$+\ r[\cos \epsilon\sqrt{1+\ell}\,(t - t(-a/\ell)) - \xi].$$

The second term is always negative.

Hence the effect of the charge separation field is to change the frequency of the motion and to slow the particle down. That is, u is slightly decreased, since ℓ is small, from its range of values without the charge separation field.

Furthermore, from (A15), at $x = 0$ we see that, if $u-1$ is close to r at $x = 0$ the ion had, at $x = -a/\ell$, $(1-u)^2 + v^2$ close to r^2.

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